The Poisson Process SS 2016
Exercise sheet 2

The exercises will be discussed in the tutorial on Thursday, May 12, 2016.

Exercise 2.1:
Let \( \eta : \Omega \to \mathbb{N}(X) \) be a mapping. Prove the equivalence of the following assertions:

(a) \( \eta \) is a point process on \( X \).
(b) \( \eta(B) \) is a random variable for each \( B \in \mathcal{X} \).

Exercise 2.2:
Let \( \gamma, \delta > 0 \).

(a) Find a random vector \((X, Y)\) such that \( X, Y \) and \( X + Y \) are Poisson distributed with parameter \( \gamma, \delta \) and \( \gamma + \delta \) respectively, but \( X \) and \( Y \) are not independent.
(b) Deduce that there exist a measure space \((X, \mathcal{X}, \lambda)\) and a point process \( \eta \) on \( X \) satisfying part (i) of the definition of a Poisson process (Definition 3.1), that is \( \eta(B) \sim \text{Po}(\lambda(B)), \ B \in \mathcal{X} \), but not part (ii), that is there exist \( m \in \mathbb{N} \) and pairwise disjoint sets \( B_1, \ldots, B_m \in \mathcal{X} \) such that \( \eta(B_1), \ldots, \eta(B_m) \) are not independent.

Exercise 2.3:
Let \( \eta \) be a Poisson process on \((X, \mathcal{X})\) with intensity measure \( \lambda \) and \( A, B \in \mathcal{X} \) with \( \lambda(A), \lambda(B) < \infty \). Show that
\[
\text{Cov}(\eta(A), \eta(B)) = \lambda(A \cap B).
\]

Exercise 2.4:
Let \( \gamma > 0 \) and \( \eta \) be a Poisson process on \( \mathbb{R}^d \) with intensity measure \( \gamma \cdot \lambda^d \). Furthermore, let
\[
d_\eta := \inf \{ \|x\| : x \in \eta \}.
\]

(a) Show that \( d_\eta \) is a random variable and determine its distribution.
(b) Let \( H \) be the cumulative distribution function of \( d_\eta \). Show that
\[
H(r) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(\eta(B(0, r)) \geq 2 \mid \eta(B(0, \varepsilon)) = 1), \quad r > 0,
\]
where \( B(0, r) \) denotes the closed ball with radius \( r \) around the origin.
Exercise 2.5:

Let $\gamma > 0$ and $\eta$ be a Poisson process on $\mathbb{R}^d$ with intensity measure $\gamma \cdot \lambda^d$. Show that

(a) $\hat{\gamma}_B := \frac{\eta(B)}{\lambda^d(B)}$ is an unbiased estimator for $\gamma$.

(b) Show that the estimator is weakly consistent in the following way: If $W_1, W_2, \ldots$ is a sequence of sets in $\mathcal{B}(\mathbb{R}^d)$ with $\lambda^d(W_n) \xrightarrow{n \to \infty} \infty$, then

$$\lim_{n \to \infty} \mathbb{P}(|\hat{\gamma}_{W_n} - \gamma| > \varepsilon) = 0, \quad \varepsilon > 0.$$  

(c) Show that the estimator is strongly consistent in the following way: If $W_1 \subset W_2 \subset \ldots$ is an increasing sequence of compact sets in $\mathcal{B}(\mathbb{R}^d)$ with $\lambda^d(W_n) \xrightarrow{n \to \infty} \infty$, then

$$\lim_{n \to \infty} \hat{\gamma}_{W_n} = \gamma \quad \mathbb{P} \text{- a.s.}$$