Seminar, Summer Semester 2023

Cohomology and Characteristic Classes

Organisers:	JProf. Dr. Manuel Krannich (krannich@kit.edu) Dr. Florian Kranhold (kranhold@kit.edu)
Time & Place:	Thu 11:30–13:00, SR −1.015
Preliminary meeting:	Thu 16 Feb 2023, 14:00, SR 3.061 If you are interested in participating, please write to krannich@kit.edu by 12 Feb 2023.
Language:	Talks can be given in English or German.
Website:	www.math.kit.edu/agt/edu/ccc2023s

For a smooth manifold M, what is the minimal number $n_M \geqslant 0$ such that M can be embedded into $\mathbb{R}^{n_M}$?

In 1944, Hassler Whitney famously proved the *upper* bound $n_M \leqslant 2 \cdot \dim(M)$ on the quantity n_M, using a certain geometric move that became later known as the *Whitney trick*. To establish *lower* bounds on n_M and to show that Whitney's bound is in fact optimal in many cases, methods from algebraic topology come to one's help, in particular the theory of *characteristic classes*.

In this seminar, we will develop the necessary tools from algebraic topology to construct one of the primary examples of characteristic classes, *Stiefel–Whitney classes*, and we will apply them to the study of smooth manifolds M, for example by giving lower bounds on the minimal embedding dimensions $n_M \geqslant 0$. We begin by introducing *cohomology*, which is a sequence of R-modules $\{H^n(X;R)\}_{n\geqslant 0}$ associated to a topological space X which only depends on the homotopy type of X; here R is a fixed commutative ring. These cohomology groups are close variants of the *homology* groups $\{H_n(X;R)\}_{n\geqslant 0}$, but have the advantage that their direct sum $\bigoplus_{n\geqslant 0} H^n(X;R)$ carries a natural ring structure, called the *cup-product*. Taking this additional structure into account can be very fruitful: For example, even though all homology and cohomology groups of $\mathbb{C}P^2$ and $S^2 \vee S^4$ are isomorphic, these spaces are not homotopy equivalent, because the ring structures on their cohomology groups differ. However, considering the cup-product alone is sometimes not enough: The cohomology rings of the once suspended spaces $\Sigma(\mathbb{C}P^2)$ and $\Sigma(S^2 \vee S^4)$, for instance, turn out to be isomorphic, but these two suspensions are still not homotopy equivalent. To distinguish them, one has to take a certain algebraic structure on the cohomology groups in addition to the ring structure into account, namely morphisms of the form $\mathrm{Sq}^i\colon H^n(X;R) \to H^{n+i}(X;R)$ for $R = \mathbb{Z}/2$, called *Steenrod squares*. After constructing Steenrod squares and discussing some of their applications, we will use them to define the aforementioned *Stiefel–Whitney classes*, which are cohomology classes $w_n(M) \in H^n(M;\mathbb{Z}/2)$ for $n \geqslant 0$, associated to a smooth manifold M. Stiefel–Whitney classes will then allow us to establish lower bounds on the minimal embedding dimensions $n_M \geqslant 0$ (for example $n_{\mathbb{R}P^4} \geqslant 8$, which is optimal by Whitney's theorem), and to draw other conclusions on the topology of smooth manifolds, such as obstructions for a manifold M to arise as the boundary of another manifold.

Prerequisites Familiarity with the content of the lecture *Algebraic Topology* (winter term 2022/23) is assumed (basic notions of homotopy and category theory, singular homology with coefficients, CW complexes, cellular homology, derived functors, the Künneth/universal coefficient theorem). If you did not attend the course but would like to take part in the seminar, please contact us.

Literature

1. G. E. Bredon. *Topology and Geometry*. Graduate Texts in Mathematics 139. Springer, 1993
2. J. W. Milnor and J. D. Stasheff. *Characteristic classes*. Annals of Mathematics Studies 76. Princeton University Press, 1974