In 1944, Hassler Whitney famously proved the upper bound \( n_M \leq 2 \cdot \dim(M) \) on the quantity \( n_M \), using a certain geometric move that became later known as the Whitney trick. To establish lower bounds on \( n_M \) and to show that Whitney’s bound is in fact optimal in many cases, methods from algebraic topology come to one’s help, in particular the theory of characteristic classes.

In this seminar, we will develop the necessary tools from algebraic topology to construct one of the primary examples of characteristic classes, Stiefel–Whitney classes, and we will apply them to the study of smooth manifolds \( M \), for example by giving lower bounds on the minimal embedding dimensions \( n_M \geq 0 \). We begin by introducing cohomology, which is a sequence of \( R \)-modules \( \{H^n(X; R)\}_{n \geq 0} \) associated to a topological space \( X \) which only depends on the homotopy type of \( X \); here \( R \) is a fixed commutative ring. These cohomology groups are close variants of the homology groups \( \{H_n(X; R)\}_{n \geq 0} \), but have the advantage that their direct sum \( \bigoplus_{n \geq 0} H^n(X; R) \) carries a natural ring structure, called the cup-product. Taking this additional structure into account can be very fruitful: For example, even though all homology and cohomology groups of \( CP^2 \) and \( S^2 \vee S^4 \) are isomorphic, these spaces are not homotopy equivalent, because the ring structures on their cohomology groups differ. However, considering the cup-product alone is sometimes not enough: The cohomology rings of the once suspended spaces \( \Sigma(CP^2) \) and \( \Sigma(S^2 \vee S^4) \), for instance, turn out to be isomorphic, but these two suspensions are still not homotopy equivalent. To distinguish them, one has to take a certain algebraic structure on the cohomology groups in addition to the ring structure into account, namely morphisms of the form \( Sq^i : H^n(X; R) \to H^{n+i}(X; R) \) for \( R = \mathbb{Z}/2 \), called Steenrod squares. After constructing Steenrod squares and discussing some of their applications, we will use them to define the aforementioned Stiefel–Whitney classes, which are cohomology classes \( w_n(M) \in H^n(M; \mathbb{Z}/2) \) for \( n \geq 0 \), associated to a smooth manifold \( M \). Stiefel–Whitney classes will then allow us to establish lower bounds on the minimal embedding dimensions \( n_M \geq 0 \) (for example \( n_{RP^1} \geq 8 \), which is optimal by Whitney’s theorem), and to draw other conclusions on the topology of smooth manifolds, such as obstructions for a manifold \( M \) to arise as the boundary of another manifold.

**Prerequisites** Familiarity with the content of the lecture Algebraic Topology (winter term 2022/23) is assumed (basic notions of homotopy and category theory, singular homology with coefficients, CW complexes, cellular homology, derived functors, the Künneth/universal coefficient theorem). If you did not attend the course but would like to take part in the seminar, please contact us.

**Literature**
