

Seminar SS 22: Vector Bundles and Topological K-theory

Language: Talks can be given in English or German.

Target audience: This seminar is aimed at Masters students in mathematics. Motivated Bachelor students in their third year may attend as well.

Prerequisites: Knowledge of some concepts from point-set topology (topological spaces, subspace and quotient topologies) and algebra (rings, ideals, polynomial rings) as taught for instance as part of the courses *Elementare Geometrie* and *Einführung in die Algebra und Zahlentheorie* will be assumed. In addition, for some of the later talks, you should be willing to familiarise yourself with constructions from linear algebra that are not always taught as part of the courses *Lineare Algebra I und II* (tensor products and exterior products). For the later talks, prior exposure to algebraic topology is beneficial, but not required since we will introduce all necessary concepts.

Time and Place: This seminar is expected to take place in person on Mondays 14:00 - 15:30, 20.30 at SR 0.019.

Registration: If you are interested in participating, please contact Manuel Krannich at

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by February 7th. The organisational meeting (“Vorbesprechung”) will take place online on

Monday, 14th February, 10:30.

The link will be distributed the week before the organisational meeting via email to those who expressed interest.

Topic of the seminar: A *division algebra structure* on \mathbb{R}^n is a bilinear map $\mu: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that the linear maps $\mu(x, -): \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\mu(-, x): \mathbb{R}^n \rightarrow \mathbb{R}^n$ are invertible for all $x \neq 0$. The multiplication maps of the real numbers \mathbb{R} , the complex number \mathbb{C} , the quaternions \mathbb{H} , and the octonions \mathbb{O} provide examples for $n = 1, 2, 4, 8$. Somewhat surprisingly, it turns out that there are no division algebra structures in any other dimension.

Although being a statement in algebra, all known proofs of this non-existence result for $n \neq 1, 2, 4, 8$ rely on concepts from topology. In this seminar we will get to know two topological tools which can be used to prove this result: *vector bundles* and *topological K-theory*. Vector bundles are a concept that mixes linear algebra—the study of vector spaces—with topology—the study of topological spaces. Informally, a *d-dimensional vector bundle* over a topological space X consists of a collection of d -dimensional vector spaces, one for every point in X , that vary continuously in X . Vector bundles over the point $X = *$ are the same as vector spaces, but if X is more complicated, then, unlike in linear algebra where two finite-dimensional vector spaces are isomorphic if they have the same dimension, there can be many different vector bundles of the same dimension over X . For example, the annulus (the product of a circle with a line) gives a 1-dimensional vector bundle over the circle and the Möbius band (the twisted product of a circle with a line) gives another one, and these are not isomorphic. *Topological K-theory* considers all vector bundles over X at once. This can help in studying topological spaces, but has also applications outside of topology, such as to the mentioned non-existence of division algebra structures on \mathbb{R}^n for $n \neq 1, 2, 4, 8$.

In this seminar, we will get to know vector bundles, learn how to extend constructions from linear algebra to vector bundles, and classify vector bundles over spheres. This will lead us to topological K-theory, which we discuss in the second part of the seminar. Along the way, we will learn about different ideas from algebraic topology.

References: We will mostly follow the first two chapters of Hatcher’s book project [3], complemented by talks that introduce some of the definitions and results Hatcher uses that might be unfamiliar. Atiyah’s lecture notes on K-theory [1] and extended seminar notes (in German) by Bratzler and Lück [2] may serve as an additional source.

[1] M. F. Atiyah. *K-theory*. Advanced Book Classics. Addison-Wesley Publishing Company, second edition, 1989.

[2] C. Bratzler and W. Lück. Topologische K-theorie. Seminar notes in German, 2017.

[3] A. Hatcher. *Vector Bundles & K-Theory*. Book project, 2017.