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A1

Global well-posedness of the Cauchy problem for the mass-subcritical NLS with initial data in a modulation space

Institute for Analysis — Workgroup Applied Analysis

CRC 1173

Wave phenomena

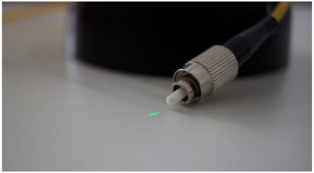
KIT – University of the State of Baden-Wuerttemberg and
National Research Center of the Helmholtz Association

http://www.waves.kit.edu/

Motivation

A Cauchy problem for the nonlinear Schrödinger equation

- Understand signal transmission in long nonlinear optical fibers.



- Governing equation is a cubic one-dimensional NLS

$$\mathrm{i}\partial_t u = -\partial_{xx} u - |u|^2 u, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}.$$

- Initial condition

$$u_0 := u(\cdot, t=0) = \sum_{n\in\mathbb{Z}} f_n(\cdot - n),$$

where f_n are selected from a finite set of functions.

- L^2-theory is not applicable, as u_0 is neither decaying nor periodic.

Motivation

Why modulation spaces?

Short-time Fourier transform w.r.t. window function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ is

$$V_g f(x,k) = \mathcal{F}(g(\cdot - x) f)(k), \qquad x,k \in \mathbb{R}^d.$$

Norm on modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ is

$$\|f\|_{M^s_{p,q}(\mathbb{R}^d)} = \left\| k \mapsto \left(1 + |k|^2\right)^{\frac{s}{2}} \|V_g f(\cdot, k)\|_p \right\|_q \quad s \in \mathbb{R},\ p,q \in [1,\infty].$$

- $u_0 \in L^\infty$, but: $e^{it\Delta} : L^p \to L^p$ iff $p = 2$.
- On the other hand: $u_0 \in M^s_{\infty,q}$ and $e^{it\Delta} : M^s_{p,q} \to M^s_{p,q}$.

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Global well-posedness result

Theorem (C.)

Cauchy problem for mass sub-critical NLS

$\mathrm{i}u_t = -\Delta u \pm |u|^{\beta-1} u \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}, \qquad u(\cdot, t=0) = u_0 \in M_{p,p'}$,

where $d \in \mathbb{N}$, $\beta \in \left(1, 1+\frac{4}{d}\right)$ *and* $p \in (2, p_{max})$ *with* $p_{max} = p_{max}(d,\beta)$, *is globally well-posed in the space*

$$C(\mathbb{R}, L^2) \cap L_{loc}^{\frac{2}{d\left(\frac{1}{2}-\frac{1}{\beta+1}\right)}}(\mathbb{R}, L^{\beta+1}) + C(\mathbb{R}, M_{\beta+1,(\beta+1)'}).$$

- Sum of spaces corresponds to high-low frequency decomposition.
- Hundertmark, Kunstmann, Pattakos and C. (2017): $d=1$, $\beta=3$, $\tilde{p}_{\max} = 2 + \frac{1}{17} < 2 + \frac{1}{3} = p_{\max}$
- These were the first global well-posedness results for the NLS with large initial data in a modulation space (which is not H^s).

Two classical Strichartz estimates

Call $(r,q) \in [2,\infty]^2$ *admissible*, if $(r,q,d) \neq (\infty,2,2)$ and

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \qquad \text{i.e.} \qquad q = q_{\mathrm{a}}(r) = \left[\frac{d}{2}\left(\frac{1}{2} - \frac{1}{r}\right)\right]^{-1}.$$

Call (ρ,γ) *dually admissible*, if (ρ',γ') are admissible, i.e. $(\rho,\gamma) \in [1,2]^2$, $(\rho,\gamma,d) \neq (1,2,2)$ and

$$\frac{2}{\gamma} + \frac{d}{\rho} = 2 + \frac{d}{2} \Leftrightarrow \gamma = \gamma_{\mathrm{a}}(\rho) = \left[1 - \frac{d}{2}\left(\frac{1}{\rho} - \frac{1}{2}\right)\right]^{-1}.$$

For admissible and dually admissible pairs one has the *homogeneous* and *inhomogeneous* Strichartz estimates

$$\left\| e^{\mathrm{i}t\Delta} u_0 \right\|_{L^{q_{\mathrm{a}}(r)}([0,T],L^r(\mathbb{R}^d))} \leq C \left\| u_0 \right\|_{L^2(\mathbb{R}^d)},$$

$$\left\| \int_0^t e^{\mathrm{i}(t-\tau)\Delta} F(\cdot,\tau)\mathrm{d}\tau \right\|_{L^{q_{\mathrm{a}}(r)}([0,T],L^r(\mathbb{R}^d))} \leq C \left\| F \right\|_{L^{\gamma_{\mathrm{a}}(\rho)}([0,T],L^{\rho}(\mathbb{R}^d))}.$$

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Mass sub-critical NLS in L^2

Tsutsumi '87

Mild solutions of the mass sub-critical NLS for IVs $v_0 \in L^2$:

$$v(\cdot,t) = e^{\mathrm{i}t\Delta}v_0 \pm \mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta}\left(|v|^{\beta-1}v\right)(\tau)\mathrm{d}\tau =: \mathcal{T}(v)(\cdot,t)$$

- Local well-posedness via Banach's fixed-point theorem in

$$M(R,T) := \left\{v \in X_1(T) \middle| \; \|v\|_{X_1(T)} \leq R\right\},$$

  where $X_1(T) := C([0,T],L^2) \cap L^{q_a(\beta+1)}([0,T],L^{\beta+1})$.
- Solution size $R \approx \|v_0\|_2$ fixed by homogeneous Strichartz estimate.
- $\left\|\int \cdots\right\|_{L^\infty(L^2)}, \left\|\int \cdots\right\|_{L^{q_a(\beta+1)}(L^{\beta+1})} \lesssim T^{1-\frac{d}{4}(\beta-1)}\|v\|^{\beta}_{L^{q_a(\beta+1)}(L^{\beta+1})}$.
- Guaranteed time of existence $T \approx \|v_0\|_2^{-a(d,\beta)}$ fixed by inhomogeneous Strichartz and Hölder's estimates.

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- Guaranteed time of existence $T \approx \|v_0\|_2^{-a(d,\beta)}$ fixed by inhomogeneous Strichartz and Hölder's estimates.
- Global existence via mass conservation $\|v(\cdot,T)\|_2 = \|v_0\|_2$.

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Initial values in $M_{p,p'}$
Local well-posedness

For $p \in (2, \beta+1)$ one has

$$M_{p,p'} = [M_{2,2}, M_{\beta+1,(\beta+1)'}]_\theta \hookrightarrow \left(L^2, M_{\beta+1,(\beta+1)'}\right)_{(\theta,\infty)} \hookrightarrow L^2 + M_{\beta+1,(\beta+1)'}.$$

It seems natural to consider the solution space

$$X(T) := C([0,T], L^2) \cap L^{q_a(\beta+1)}([0,T], L^{\beta+1}) + C([0,T], M_{\beta+1,(\beta+1)'}).$$

Contraction operator $\mathcal{T}$ for $u_0 = v_0 + w_0$ is given by

$$\mathcal{T}(u) = \underbrace{e^{it\Delta} w_0}_{M_{\beta+1,(\beta+1)'}} + \underbrace{e^{it\Delta} v_0 \pm i \int_0^t e^{i(t-\tau)\Delta} \left(|u|^{\beta-1} u\right)(\tau) d\tau}_{L^2}.$$

- $M_{\beta+1,(\beta+1)'} \hookrightarrow L^{\beta+1}$, so $X \hookrightarrow L^{q_a(\beta+1)}(L^{\beta+1})$.
- As in the L^2 case: $R \approx \|u_0\|$ and $T \approx \|u_0\|^{-\frac{1}{\frac{1}{\beta-1} - \frac{d}{4}}}$

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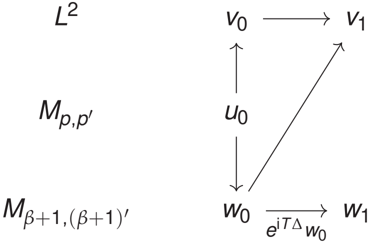
Global well-posedness

Salvaging mass conservation



Global well-posedness

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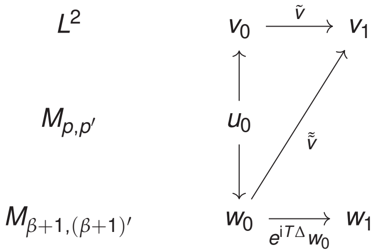


Let $\tilde{v}$ solve the NLS with IV v_0. Refined splitting ansatz

$$u = \tilde{v} + \tilde{\tilde{v}} + e^{it\Delta} w_0$$

leads to a fixed-point equation for $\tilde{\tilde{v}}$. Banach's contraction mapping works for

- $T \approx \min \left\{ \left(\|v_0\|_2 + \|w_0\|_{M_{\beta+1,(\beta+1)'}} \right)^{-a(d,\beta)}, \|w_0\|_{M_{\beta+1,(\beta+1)'}}^{-b(d,\beta)} \right\}$,
- $\left\| \tilde{\tilde{v}}(\cdot, T) \right\|_2 \lesssim R \approx T^{c(d,\beta)} \|w_0\|_{M_{\beta+1,(\beta+1)'}}$.

Global well-posedness

Norms of the IVs and reiteration

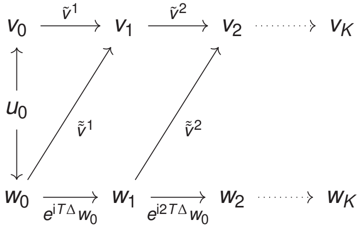
Interpolation theory: there is an $\alpha = \alpha(p, \beta)$ such that for any $N > 0$

$$u_0 = v_0 + w_0, \qquad \|v_0\|_2 \lesssim N^\alpha, \qquad \|w_0\|_{M_{\beta+1,(\beta+1)'}} \lesssim \frac{1}{N}.$$

Fix timestep "$T = 2N^{-\alpha a(d,\beta)}$" and reiterate, as often as possible.

$\|\cdot\|_2 \lesssim 2N^\alpha$

$\|\cdot\|_{M_{\beta+1,(\beta+1)'}} \lesssim \frac{1}{N}$



- On bounded time intervals: $\|w_j\| \lesssim \frac{1}{N}$.
- Solve failure condition $\|v_K\|_2 > 2N^\alpha$ for $KT \gtrsim N^{a'(\alpha,d,\beta)} \xrightarrow{N\to\infty} \infty$.

Related works

Global well-posedness results

Modulation spaces

- Oh, T. and Wang, Y. (2018, preprint): $d = 1$, $\beta = 3$ in $M_{2,q}$ for $q < \infty$. Use conserved quantities constructed by Killip-Vișan-Zhang.
- Wang, B. and Hudzik, H. (2007), also Kato, T. (2014): Use Strichartz-type estimates in modulation space $M_{2,1}$. Require small initial data, do not cover $d = 1$, $\beta = 3$.

Other spaces outside L^2

- Hyakuna, R. and Tsustumi, M. (2012): $u_0 \in \widehat{L}^p$ with p sufficiently close to 2. Use HLFD. Our main inspiration.
- Grünrock, A. (2005): $u_0 \in \widehat{L}^p$ with p sufficiently close to 2. LWP via the Fourier norm restriction method, GWP via HLFD.
- Vargas A. and Vega L. (2001): $u_0 \in L^2 + Y_{3,6}$ with trading exponent $\alpha < 1$. Also HLFD.

Related works

Local well-posedness results

Modulation spaces

- Sugimoto, S. and Tomita, N and Wang, B. (2011). See also C. (2018): $M^s_{p,q} \cap M_{\infty,1}$ for $s \geq 0$ are Banach *-algebras $\rightsquigarrow$ algebraic nonlinearities. Special cases in Feichtinger (1983); Wang, B. and Zhao, L. and Guo, B. (2006); Bényi, A. and Okoudjou, K. (2009).
- Guo, S. (2017): LWP for $d = 1$, $\beta = 3$ in $M_{2,q}$ for $q \in [2, \infty)$ via U^p, V^p spaces.
- Hundertmark, D. and Kunstmann, P. and Pattakos, N. and C. (2019, JEE): LWP for $d = 1$, $\beta = 3$ in $M^s_{p,q}$ for some ranges of p, q and s via the differentiation by parts technique.

More recent research

Tooth problems

Motivation

- Global existence in $M^s_{\infty,q}$ seems hard (no known conserved quantities).
- Easier problem: $u_0 \in H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})$.
- Think of the first summand cancelling finitely many *teeth* of the second summand ($s_1 \leq s_2$).

Results

- HKPC (2018, preprint): LWP for $d = 1$, $\beta = 3$ in $H^s(\mathbb{R}) + H^{\frac{1}{2}+\varepsilon}(\mathbb{T})$ for $s \in \left[\frac{1}{6}, \frac{1}{2}\right]$ via the differentiation by parts technique.
- HKPC (2019, preprint): GWP for $d = 1$, $\beta = 2$ in $L^2(\mathbb{R}) + H^1(\mathbb{T})$ via Strichartz estimates and Gronwall's lemma.

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Thank you for your attention!

Modulation spaces

An equivalent norm

Consider a smooth partition of unity $(\sigma_k)_{k\in\mathbb{Z}^d} \subseteq C^\infty$ satisfying

- $|\sigma_k(\xi)| \geq c$ for all $\xi \in Q_k := k + \left[-\frac{1}{2}, \frac{1}{2}\right)^d$,
- $\operatorname{supp}(\sigma_k) \subseteq B_{\sqrt{d}}(k)$,
- $\|\partial^\alpha \sigma_k\|_\infty \leq C_\alpha$ independently of k.

Define *isometric decomposition operators* $\square_k := \mathcal{F}^{(-1)}\sigma_k\mathcal{F}$. Then

$$u \mapsto \left\| \left(\langle k \rangle^s \|\square_k u\|_p \right)_{k\in\mathbb{Z}^d} \right\|_q$$

is an equivalent norm for $M^s_{p,q}$.

Multiplier estimate

Moreover,

$$\|\square_k\|_{\mathscr{L}(L^p(\mathbb{R}^d))} \leq C \qquad \forall k \in \mathbb{Z}^d.$$

Modulation spaces

Basic properties

- $M^s_{p,q} \subseteq \mathcal{S}'$ are Banach spaces.
- For $s_1 \leq s_2$, $p_1 \leq p_2$ and $q_1 \leq q_2$ one has
$$M^{s_2}_{p_1,q_1} \subseteq M^{s_1}_{p_2,q_2}.$$
- One has $\mathcal{S} \subseteq M^s_{p,q}$. If $p, q < \infty$ then $\mathcal{S}$ is even dense in $M^s_{p,q}$.
- For $p, q < \infty$, one has $(M^s_{p,q})' = M^{-s}_{p',q'}$ and (extension of L^2-duality)
$$\langle v, u \rangle = \sum_{|l| \leq \frac{3}{2}\sqrt{d}} \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \overline{\square_{k+l} v} \square_k u \qquad u \in M^s_{p,q},\ v \in M^{-s}_{p',q'}.$$
- Let $p_1, q_1 \in [1, \infty)$, $p_2, q_2 \in [1, \infty]$, $s_1, s_2 \in \mathbb{R}$ and $\theta \in (0, 1)$. Then
$$[M^{s_1}_{p_1,q_1}, M^{s_2}_{p_2,q_2}]_\theta = M^s_{p,q} \qquad \text{(complex interpolation), if}$$
$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \qquad s = (1-\theta)s_1 + \theta s_2.$$

Improved local well-posedness

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Theorem (Dirk Hundertmark, Peer Kunstmann, Nikolaos Pattakos and C.)

The Cauchy problem for the NLS with an algebraic nonlinearity

$$\mathrm{i}u_t = -\Delta u + F(|u|^2)u \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}, \quad u(\cdot, t=0) = u_0 \in M^s_{p,q} \cap M_{\infty,1},$$

where $F(z) = \sum_{k=1}^{\infty} a_k z^k$ *entire function and* $s \geq 0$*, is locally well-posed in* $C([0,T], M^s_{p,q} \cap M_{\infty,1})$.

- $M^s_{p,q} \hookrightarrow M_{\infty,1} \hookrightarrow C_{\mathrm{b}}$ for $q=1$ and $s \geq 0$ or $q > 1$ and $s > \frac{d}{q'}$.
- $M_{2,1}$: Wang, Zhao and Guo in '06.
- $M_{p,1}$ for $p \in [1, \infty]$: Bényi and Okoudjou in '09.
- Proof: boundedness of the Schrödinger propagator and algebra property of $M^s_{p,q} \cap M_{\infty,1}$.

Improved local well-posedness

Structure of the proof

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Interested in mild solutions, i.e.

$$u(\cdot, t) = e^{\mathrm{i}t\Delta}u_0 + \int_0^t e^{\mathrm{i}(t-\tau)\Delta}(F(|u|^2)u)(\tau)\mathrm{d}\tau =: \mathcal{T}(u).$$

By Banach's fixed-point theorem, it is enough to show that $\mathcal{T}$ is a contraction on

$$M(R, T) := \{u \in C([0, T], X) \,|\, \forall \tau \in [0, T] : \|u(\tau)\|_X \leq R\}$$

for some $R, T > 0$, where $X := M^s_{p,q} \cap M_{\infty,1}$.

- Schrödinger propagator is polynomially bounded on X.
- X is a *Banach *-algebra*, hence $u \mapsto F(|u|^2)u$ is locally Lipschitz.

Algebra property of $M^s_{p,q} \cap M_{\infty,1}$

$M^s_{p,q}$ is a Banach *-algebra, if $q = 1$ and $s \geq 0$ or $q > 1$ and $s > \frac{d}{q'}$.

- Shown first by Feichtinger in '83.
- Again for $M_{2,1}$ by Wang et al. in '06 and for $M_{p,1}$ by Bényi et al. in '09.
- For $M^s_{p,q} \cap M_{\infty,1}$, use an estimate as in Sugimoto, Tomita and Wang (2011).

$$
\begin{aligned}
\|uv\|_{M^s_{p,q}} &= \left\| \left(\langle k \rangle^s \|\Box_k uv\|_p \right)_{k \in \mathbb{Z}^d} \right\|_q, \\
\langle k \rangle^s &\lesssim \langle k-m \rangle^s + \langle m \rangle^s \lesssim \langle k+l-m \rangle^s + \langle m \rangle^s, \\
\mathcal{F}\Box_k(uv) &= \sigma_k \sum_{l,m} (\sigma_{l-m}\hat{u}) * (\sigma_m \hat{v}) \\
&= \sigma_k \sum_{l \in \Lambda} \sum_m (\sigma_{k+l-m}\hat{u}) * (\sigma_m \hat{v}), \\
\Rightarrow \langle k \rangle^s \|\Box_k(uv)\|_p &\lesssim \sum_{l \in \Lambda} \sum_m \langle k+l-m \rangle^s \|\Box_{k+l-m}u \Box_m v\|_p + \\
&\quad \sum_{l \in \Lambda} \sum_m \langle m \rangle^s \|\Box_{k+l-m}u\|_\infty \|\Box_m v\|_p .
\end{aligned}
$$

$$p_{\max} = \begin{cases} 2 + \frac{2}{\beta} - \frac{d}{2}\left(1 - \frac{1}{\beta}\right) & \text{if } \beta > \frac{1}{2} - \frac{d}{4} + \sqrt{2 + \left(\frac{1}{2} + \frac{d}{4}\right)^2}, \\ \beta + 1 & \text{otherwise,} \end{cases}$$