

Mantelflächen schiefer Körper

CAS-Maple-Tagung
Karlsruher Institut für Technologie (KIT)
28. Februar 2012

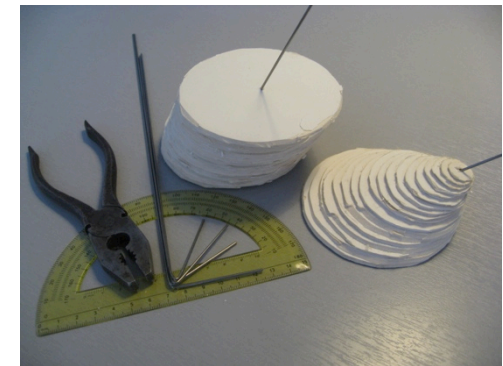
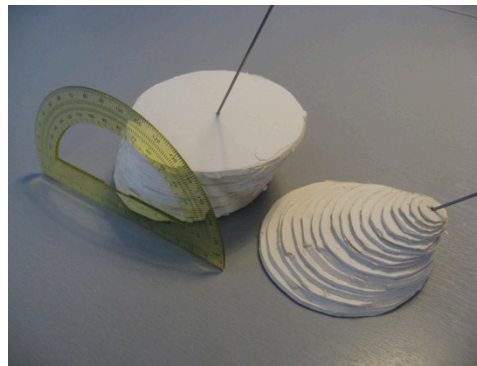
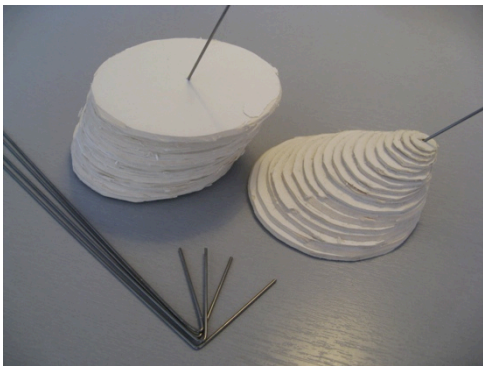
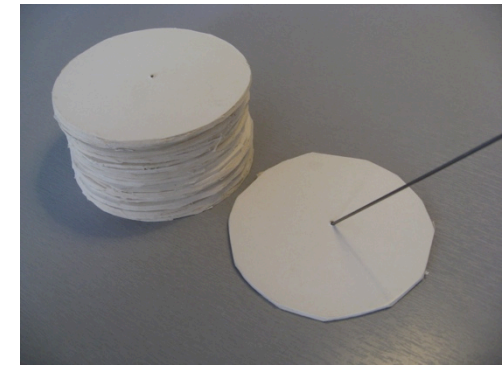
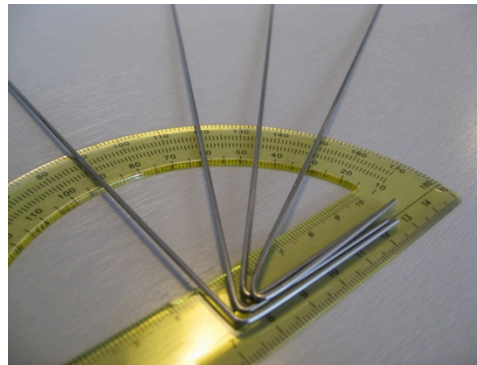
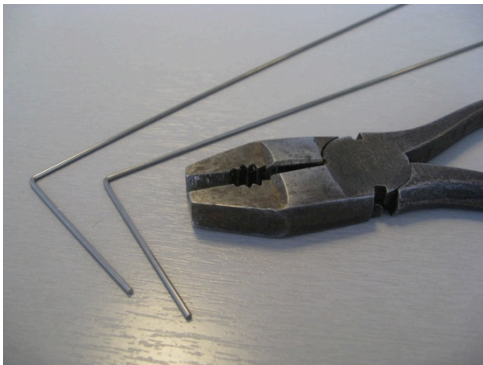
StR Martin Renner
Markgrafengymnasium, Gymnasiumstr. 1–3, 76227 Karlsruhe

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-

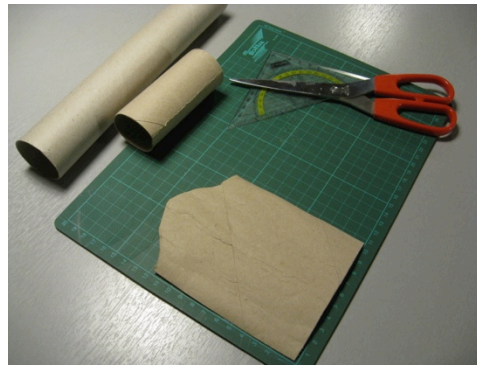
Praxis – Modelle

■ Kreisscheiben



Praxis – Modelle

▣ Papprollen



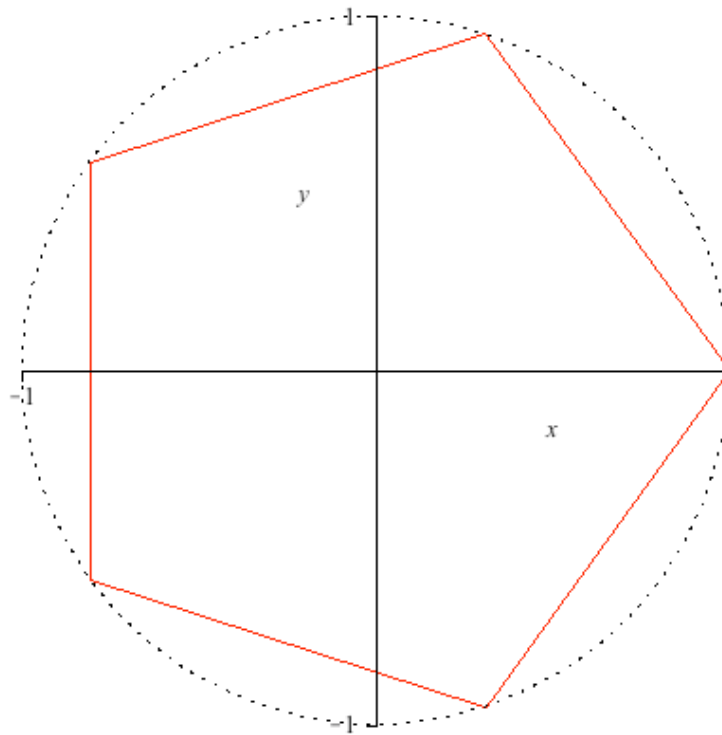
▣ Verpackungen

▣ Styroporkörper

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Theorie – Prisma und Zylinder

- Gerades Prisma
 - Grundfläche als regelmäßiges Vieleck ($n = 5$)



$$\vec{x}_i \Big|_{i=0}^{n-1} = \begin{pmatrix} r \cdot \cos(2i\pi/n) \\ r \cdot \sin(2i\pi/n) \\ z \end{pmatrix}$$

Theorie – Prisma und Zylinder

- Gerades Prisma
 - Grundfläche als regelmäßiges Vieleck

```
> restart;
> r:=1;
> h:=2;|
                                     r:=1
                                     h:=2

> n:=5;
                                     n:=5

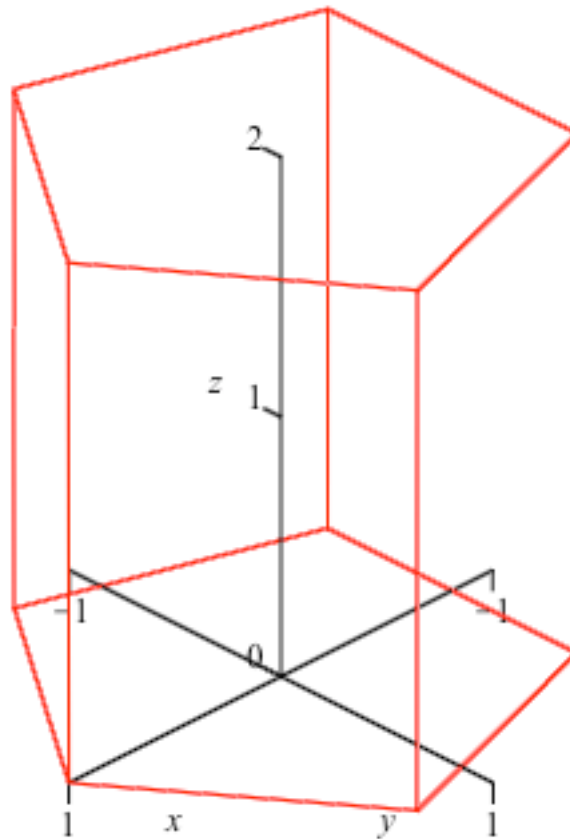
> EG:=seq([r*cos(2*i*Pi/n),r*sin(2*i*Pi/n),0],i=0..n-1)];
evalf(EG);
EG := [[1, 0, 0], [cos(2/5 π), sin(2/5 π), 0], [-cos(1/5 π), sin(1/5 π), 0], [-cos(1/5 π), -sin(1/5 π), 0],
[cos(2/5 π), -sin(2/5 π), 0]]
[[1., 0., 0.], [0.3090169938, 0.9510565165, 0.], [-0.8090169943, 0.5877852524, 0.], [-0.8090169943, -0.5877852524,
0.], [0.3090169938, -0.9510565165, 0.]]

> ED:=seq([r*cos(2*i*Pi/n),r*sin(2*i*Pi/n),h],i=0..n-1)];
evalf(ED);
ED := [[1, 0, 2], [cos(2/5 π), sin(2/5 π), 2], [-cos(1/5 π), sin(1/5 π), 2], [-cos(1/5 π), -sin(1/5 π), 2],
[cos(2/5 π), -sin(2/5 π), 2]]
[[1., 0., 2.], [0.3090169938, 0.9510565165, 2.], [-0.8090169943, 0.5877852524, 2.], [-0.8090169943, -0.5877852524,
2.], [0.3090169938, -0.9510565165, 2.]]
```

Theorie – Prisma und Zylinder

- Gerades Prisma

- Kantenmodell

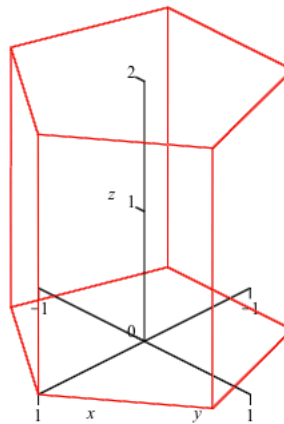


Theorie – Prisma und Zylinder

■ Gerades Prisma

■ Kantenmodell

```
> plots[display](  
  plottools[polygon](EG,color=red,style=line),  
  plottools[polygon](ED,color=red,style=line),  
  seq(plottools[line](EG[i],ED[i],color=red),i=1..n),  
  axes=normal,orientation=[45,60,0],scaling=constrained);
```



Theorie – Prisma und Zylinder

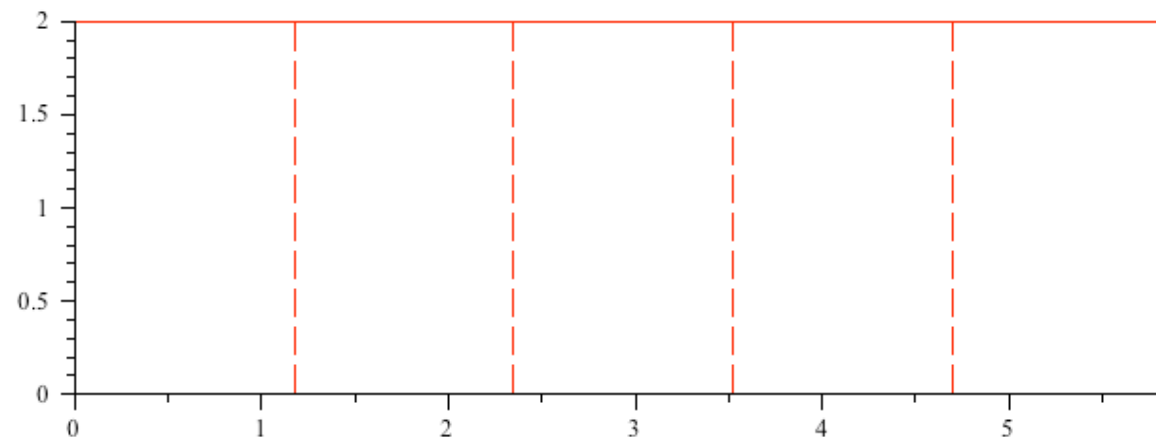
- Gerades Prisma
 - Kantenlänge a, Umfangslänge U = na
 - Mantelfläche

```
> a:=sqrt((EG[1][1]-EG[2][1])^2+(EG[1][2]-EG[2][2])^2);  
evalf(%);
```

$$a := \sqrt{\left(1 - \cos\left(\frac{2}{5}\pi\right)\right)^2 + \sin\left(\frac{2}{5}\pi\right)^2}$$

1.175570505

```
> plots[display](  
  plottools[polygon]([0,0],[n*a,0],[n*a,h],[0,h],color=red,style=line),  
  seq(plottools[line]([i*a,0],[i*a,h],color=red,linestyle=3),i=1..n-1),  
  scaling=constrained);
```



Theorie – Prisma und Zylinder

■ Gerader Zylinder

■ Parametrisierung

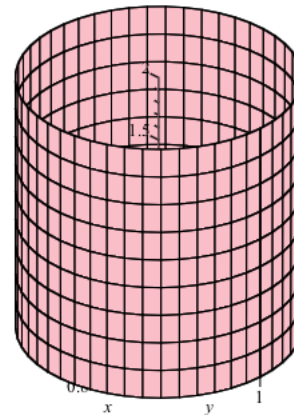
$$Z : \vec{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} = \begin{pmatrix} r \cdot \sin(u) \\ r \cdot \cos(u) \\ v \end{pmatrix}$$

Theorie – Prisma und Zylinder

■ Gerader Zylinder

■ 3D-Modell

```
> r:=1;  
h:=2;  
  
r:=1  
h:=2  
  
> Z:=(u,v)->[r*sin(u),r*cos(u),v]; Z(u,v);  
Z:=(u,v) -> [r sin(u), r cos(u), v]  
[sin(u), cos(u), v]  
  
> plot3d(Z(u,v),u=0..2*Pi,v=0..h,axes=normal,color=pink,grid=[50,10],  
orientation=[45,60,0],scaling=constrained);
```

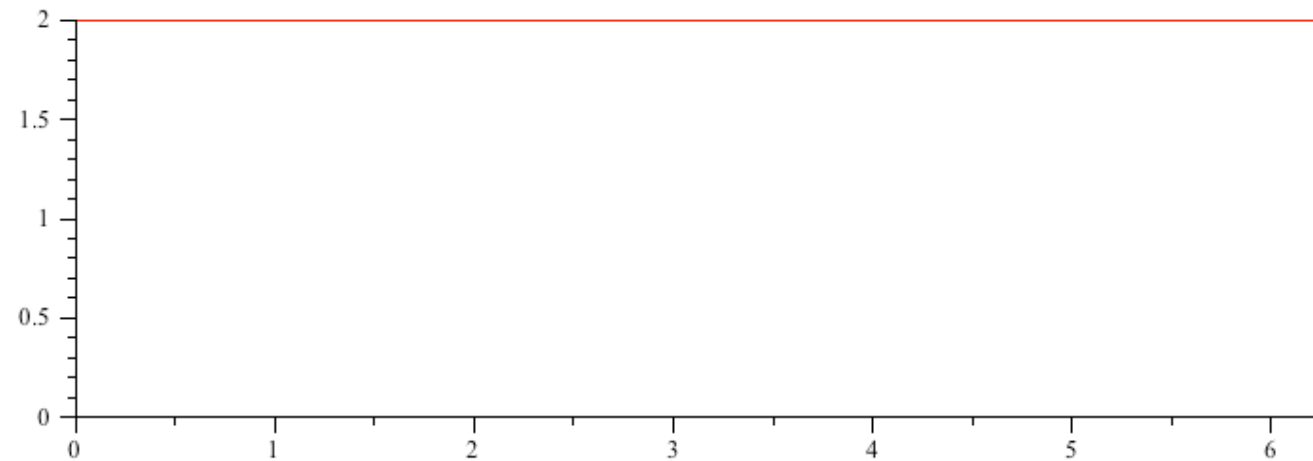


Theorie – Prisma und Zylinder

■ Gerader Zylinder

■ Mantelfläche

```
[> plots[display](  
  plottools[polygon]([ [0,0], [2*Pi*r,0], [2*Pi*r,h], [0,h] ], color=red, style=line),  
  scaling=constrained);
```

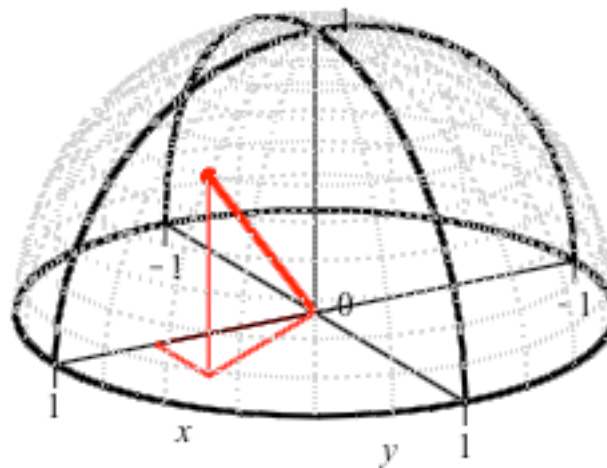


Theorie – Prisma und Zylinder

■ Schiefes Prisma

■ Scherung parallel zur xy-Ebene (Grundrissebene)

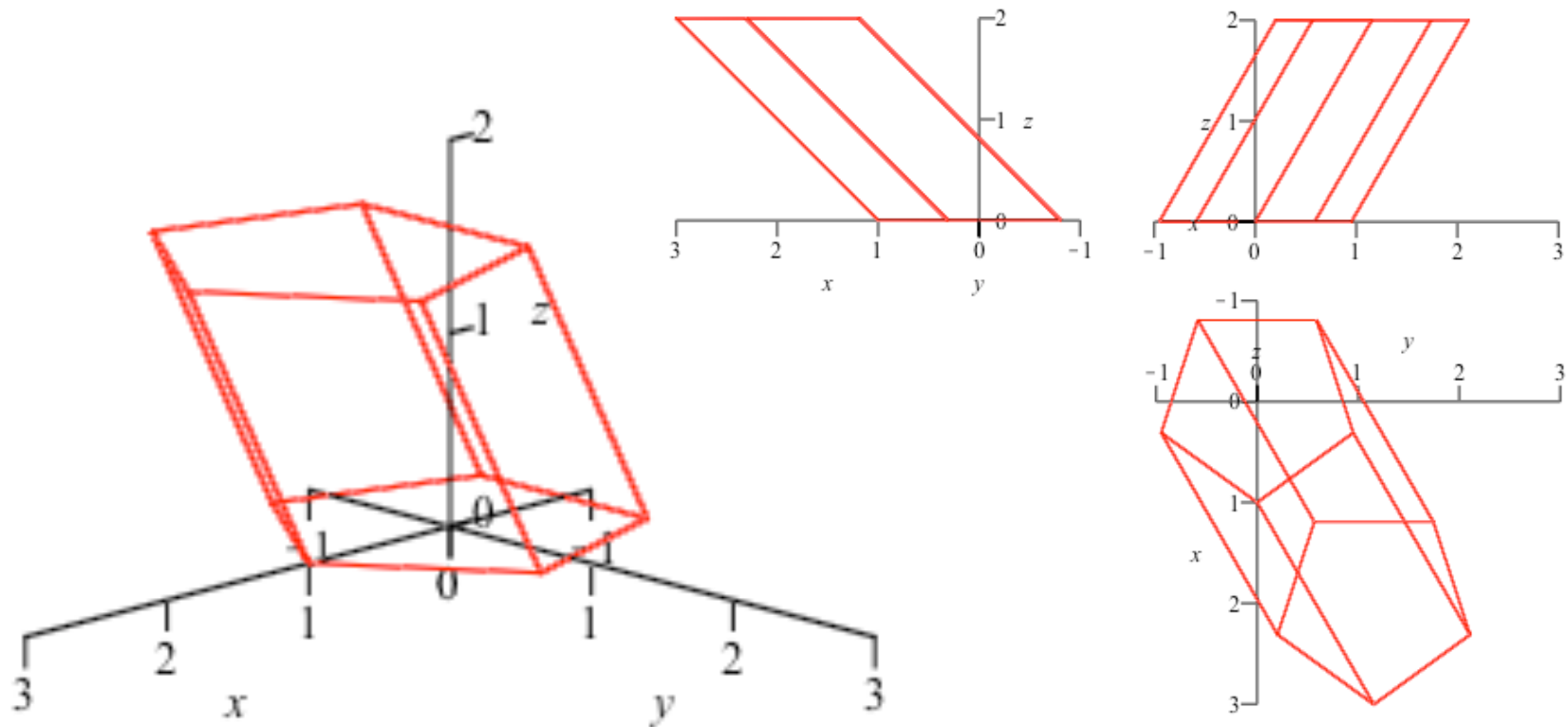
$$S : \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \vec{x}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + a \cdot z \\ y + b \cdot z \\ z \end{pmatrix} = \begin{pmatrix} x + \cot(\varphi) \cdot z \\ y + \tan(\vartheta) \cdot z \\ z \end{pmatrix}$$



Theorie – Prisma und Zylinder

■ Schiefes Prisma

■ Kantenmodell ($n = 5$, $\varphi = 45^\circ$, $\vartheta = 30^\circ$)



Theorie – Prisma und Zylinder

■ Schiefes Prisma

■ Kantenmodell

```
> r:=1;  
  h:=2;  
  
-  
> n:=5;  
  
-  
> phi:=45;  
  
-  
> phi:=convert(phi*degrees,radians);  
  
-  
> theta:=30;  
  
-  
> theta:=convert(theta*degrees,radians);
```

$r := 1$
$h := 2$
$n := 5$
$\phi := 45$
$\phi := \frac{1}{4} \pi$
$\theta := 30$
$\theta := \frac{1}{6} \pi$

Theorie – Prisma und Zylinder

■ Schiefes Prisma

■ Kantenmodell

```
> ED:= [seq([r*cos(2*i*Pi/n)+h*cot(phi), r*sin(2*i*Pi/n)+h*tan(theta), h], i=0..n-1)];  
evalf(ED);  
ED:= [[3, 2/3*sqrt(3), 2], [cos(2/5*Pi)+2, sin(2/5*Pi)+2/3*sqrt(3), 2], [-cos(1/5*Pi)+2, sin(1/5*Pi)+2/3*sqrt(3), 2],  
      [-cos(1/5*Pi)+2, -sin(1/5*Pi)+2/3*sqrt(3), 2], [cos(2/5*Pi)+2, -sin(2/5*Pi)+2/3*sqrt(3), 2]]  
[[3., 1.154700539, 2.], [2.309016994, 2.105757056, 2.], [1.190983006, 1.742485791, 2.], [1.190983006, 0.5669152866,  
2.], [2.309016994, 0.2036440225, 2.]]  
> plots[display](  
  plottools[polygon](EG, color=red, style=line),  
  plottools[polygon](ED, color=red, style=line),  
  seq(plottools[line](EG[i], ED[i], color=red), i=1..n),  
  axes=normal, orientation=[45, 60, 0], scaling=constrained);
```

Theorie – Prisma und Zylinder

■ Schiefes Prisma

■ Kantenlänge a der Grundfläche

```
> a:=sqrt((EG[1][1]-EG[2][1])^2+(EG[1][2]-EG[2][2])^2);  
evalf(%);
```

$$a := \sqrt{\left(1 - \cos\left(\frac{2}{5}\pi\right)\right)^2 + \sin\left(\frac{2}{5}\pi\right)^2}$$

1.175570505

■ Winkel α der Seitenflächenparallelogramme (Skalarprodukt der Grund- und Seitenkantenvektoren)

$$\cos(\alpha_i) = \frac{\vec{a}_i \cdot \vec{s}_i}{|\vec{a}_i| \cdot |\vec{s}_i|}$$

```
> with(linalg):  
> for i from 1 to n-1 do  
  alpha[i]:=arccos(simplify(dotprod(normalize(EG[i+1]-EG[i]),normalize(ED[i]-EG[i]))))  
  ;  
  evalf(convert(alpha[i],degrees));  
od;  
alpha[n]:=arccos(simplify(dotprod(normalize(EG[1]-EG[n]),normalize(ED[n]-EG[n]))));  
evalf(convert(alpha[n],degrees));
```

Theorie – Prisma und Zylinder

■ Schiefes Prisma

■ Winkel α der Seitenflächenparallelogramme

$$\left[\begin{array}{l} \alpha_1 := 94.53201286 \text{ degrees} \\ \alpha_2 := 137.6811881 \text{ degrees} \\ \alpha_3 := 112.2076543 \text{ degrees} \\ \alpha_4 := 59.61451828 \text{ degrees} \\ \alpha_5 := 46.32432161 \text{ degrees} \end{array} \right.$$

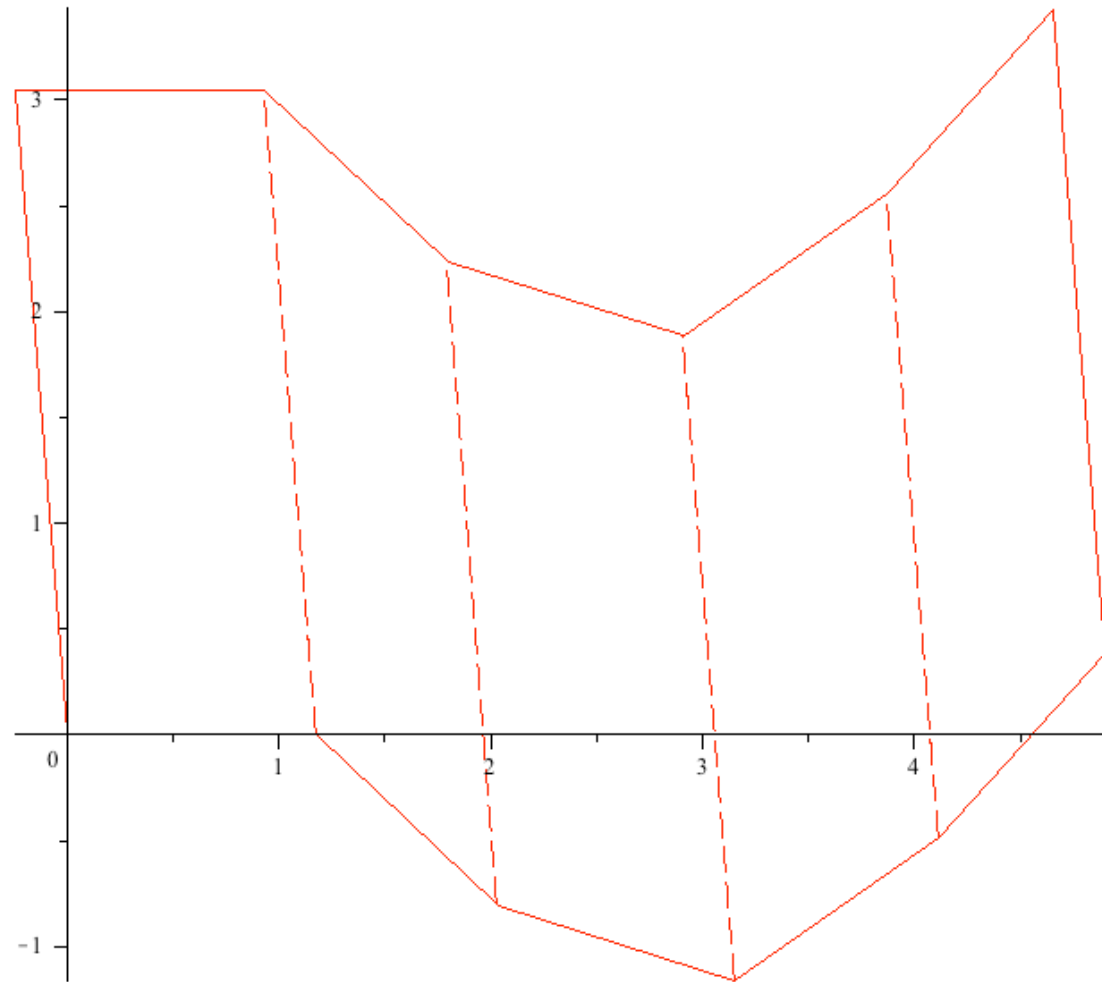
■ Höhe der Mantelfläche

$$\left[\begin{array}{l} > \text{d} := (X, Y) \rightarrow \sqrt{(X[1] - Y[1])^2 + (X[2] - Y[2])^2 + (X[3] - Y[3])^2}; \\ & d := (X, Y) \rightarrow \sqrt{(X_1 - Y_1)^2 + (X_2 - Y_2)^2 + (X_3 - Y_3)^2} \\ > \text{h_A} := d(\text{EG}[1], \text{ED}[1]); \text{evalf}(\%); \\ & h_A := \frac{2}{3} \sqrt{21} \\ & 3.055050463 \end{array} \right.$$

Theorie – Prisma und Zylinder

■ Schiefes Prisma

■ Mantelfläche



Theorie – Prisma und Zylinder

■ Schiefes Prisma

■ Mantelfläche

```
> # untere Mantellinie
MLu:=[ [0,0],[a,0]]:
for i from 2 to n do
  MLu:=[op(MLu),[MLu[i][1]+a*cos(alpha[1]-alpha[i]),MLu[i][2]+a*sin(alpha[1]-alpha[i])
  ]];
od:
print(evalf(MLu));
[[0., 0.], [1.175570505, 0.], [2.033238031, -0.8039729015], [3.153310618, -1.160909048], [4.117251558,
-0.4880168755], [4.900689823, 0.3884478969]]

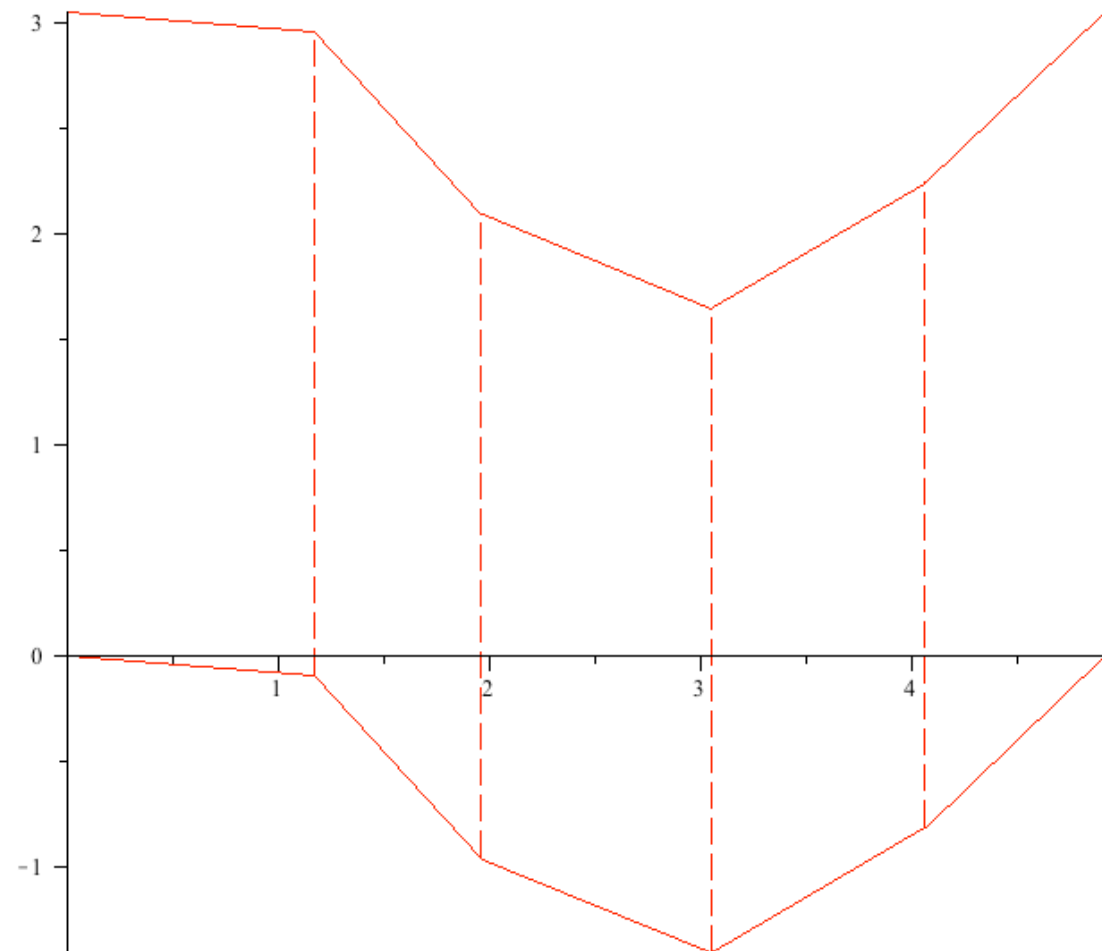
> # obere Mantellinie
MLo:=[]:
for i from 1 to n+1 do
  MLo:=[ [MLu[i][1]+h_A*cos(alpha[1]),MLu[i][2]+h_A*sin(alpha[1])],op(MLo)];
od:
print(evalf(MLo));
[[4.659291677, 3.433946259], [3.875853412, 2.557481486], [2.911912472, 1.884589314], [1.791839885, 2.241525460],
[0.9341723587, 3.045498362], [-0.2413981463, 3.045498362]]

> plots[display](
  plottools[polygon]([op(MLu),op(MLo)],color=red,style=line),
  seq(plottools[line](MLu[i],MLo[n+2-i],color=red,linestyle=3),i=2..n),
  scaling=constrained);
```

Theorie – Prisma und Zylinder

■ Schiefes Prisma

■ Mantelfläche



Theorie – Prisma und Zylinder

■ Schiefes Prisma

■ Mantelfläche

```
> # untere Mantellinie
MLu:=[ [0,0], [a*cos(Pi/2-alpha[1]), a*sin(Pi/2-alpha[1])] ]:
for i from 2 to n do
MLu:=[op(MLu), [MLu[i][1]+a*cos(Pi/2-alpha[i]), MLu[i][2]+a*sin(Pi/2-alpha[i])] ]:
od:
print(evalf(MLu));
[[0., 0.], [1.171894897, -0.09288898641], [1.963353994, -0.9621176737], [3.051720796, -1.406441560],
[4.065817131, -0.8118201370], [4.916060672, -3. 10-10]]

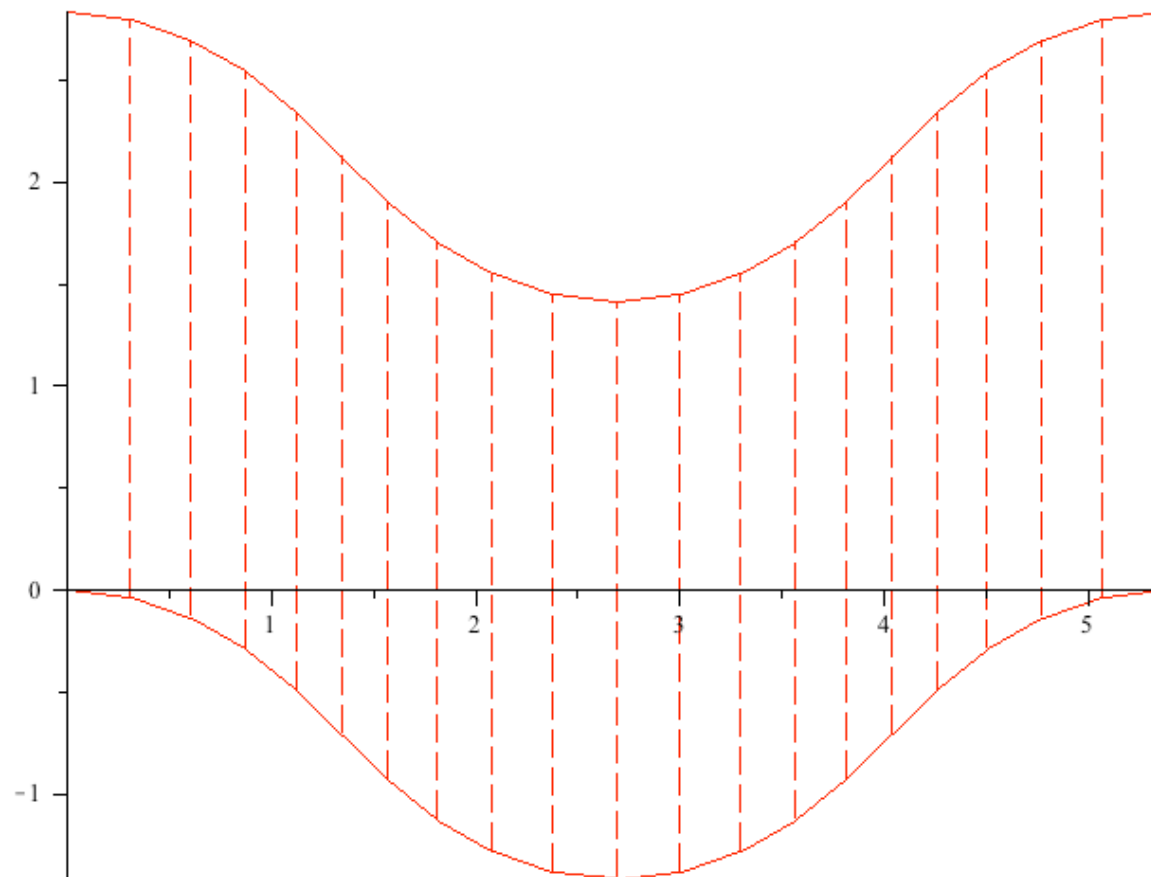
> # obere Mantellinie
MLo:=[]:
for i from 1 to n+1 do
MLo:=[ [MLu[i][1], MLu[i][2]+h_A], op(MLo) ]:
od:
print(evalf(MLo));
[[4.916060672, 3.055050463], [4.065817131, 2.243230326], [3.051720796, 1.648608903], [1.963353994, 2.092932789],
[1.171894897, 2.962161477], [0., 3.055050463]]

> plots[display](
plottools[polygon]([op(MLu), op(MLo)], color=red, style=line),
seq(plottools[line](MLu[i], MLo[n+2-i], color=red, linestyle=3), i=2..n),
scaling=constrained);
```

Theorie – Prisma und Zylinder

■ Schiefer Zylinder

■ Näherungslösung (schiefes Prisma für $n = 20$, $\varphi = 45^\circ$, $\vartheta = 0^\circ$)



Theorie – Prisma und Zylinder

Schiefer Zylinder

■ Parametrisierung

$$Z : \vec{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} = \begin{pmatrix} r \cdot \sin(u) + v \cdot \cot(\varphi) \\ r \cdot \cos(u) \\ v \end{pmatrix}$$

```
> r:=1;
h:=2;

r:=1
h:=2

> phi:=45;

phi:=45

> phi:=convert(phi*degrees,radians);

phi:=1/4 pi

> Z:=(u,v)->[r*sin(u)+v/sin(phi)*cos(phi),r*cos(u),v]; Z(u,v);

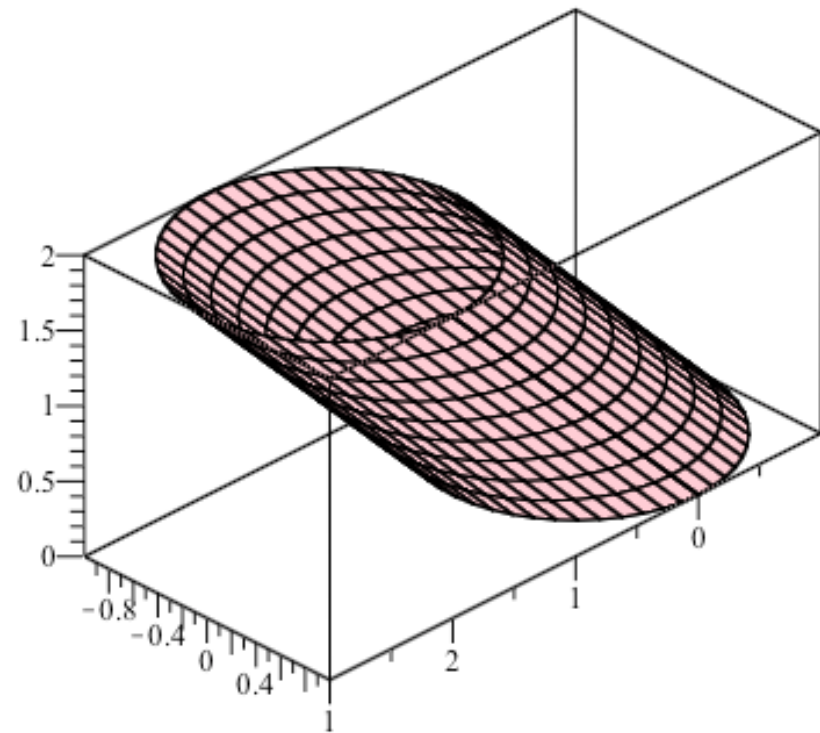
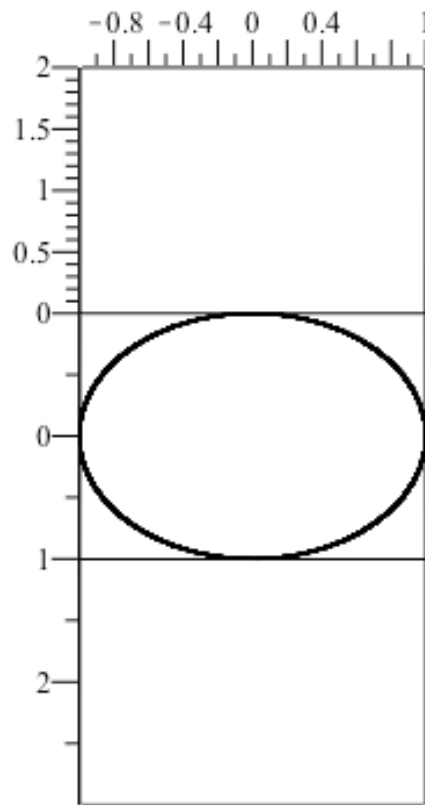
Z:=(u,v) -> [r sin(u) + v cos(phi)/sin(phi), r cos(u), v]

[sin(u) + v, cos(u), v]
```

Theorie – Prisma und Zylinder

■ Schiefer Zylinder

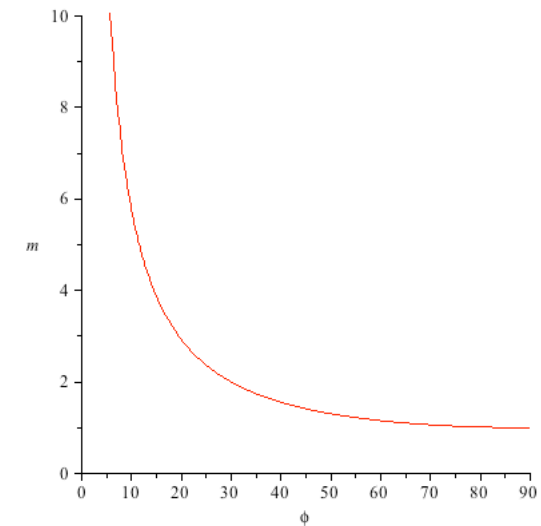
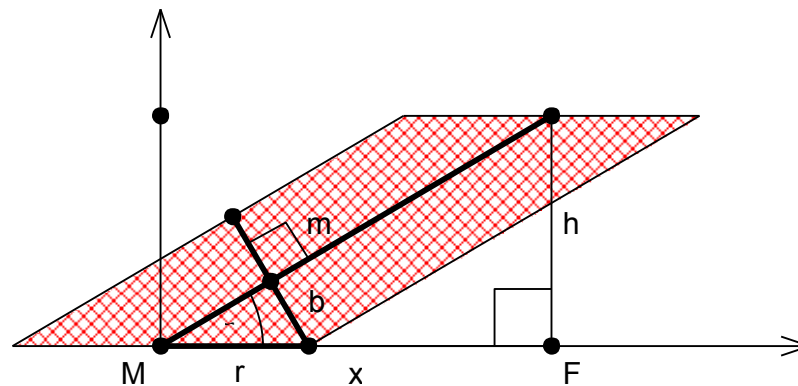
■ 3D-Modell



Theorie – Prisma und Zylinder

■ Schiefer Zylinder

■ Mantellinie m



■ Abrollkurve

Ellipse mit Halbachsen $a = r$ und $b = r \cdot \sin(\varphi)$

```
> a:=r;  
  b:=r*sin(phi); evalf(b);
```

$$a := 1$$

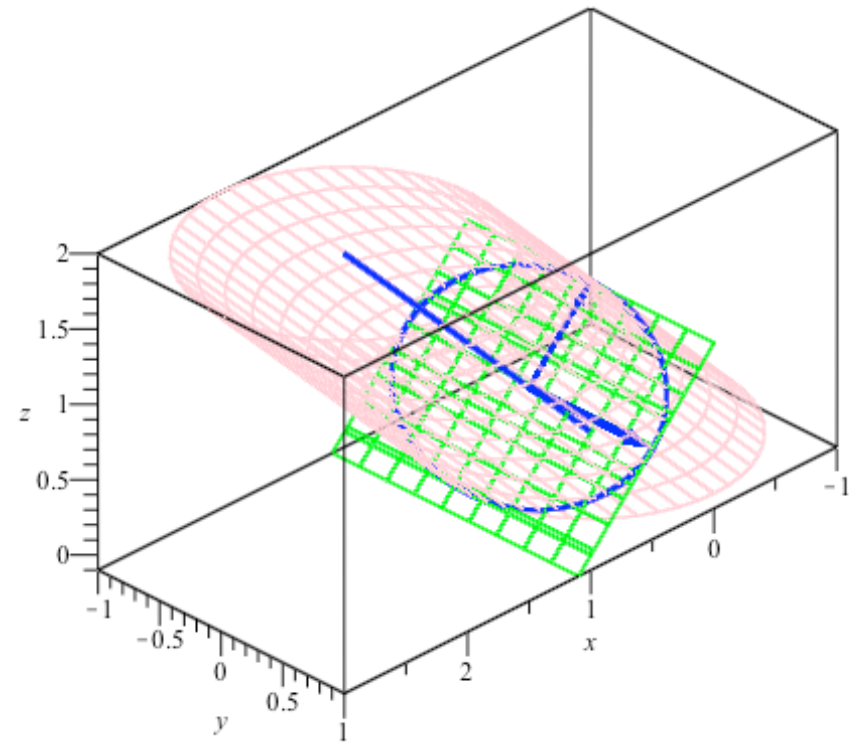
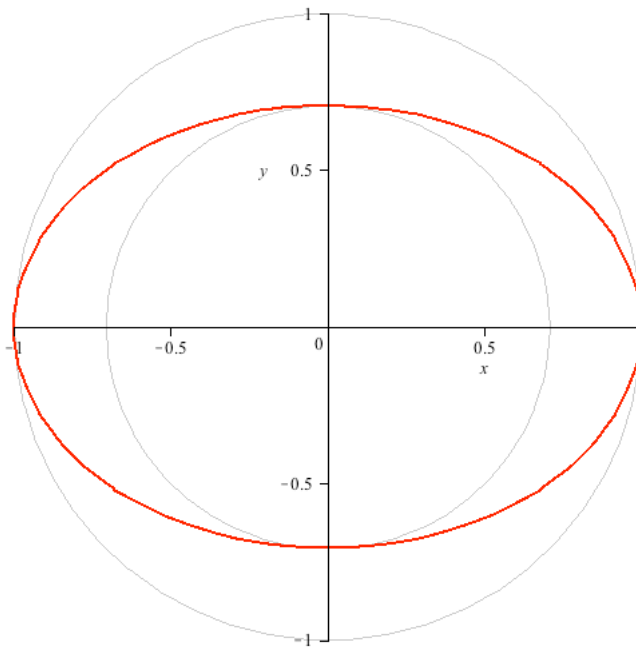
$$b := \frac{1}{2}\sqrt{2}$$

$$0.7071067810$$

Theorie – Prisma und Zylinder

■ Schiefer Zylinder

■ Abrollkurve



Theorie – Prisma und Zylinder

- Schiefer Zylinder

- Bogenlänge (funktional)

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx$$

Bogenlänge (parametrisiert)

$$L = \int_a^b |\vec{x}'(t)| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$$

- Die Umfanglänge einer Ellipse ist im allgemeinen nicht durch elementare Funktionen darstellbar, sondern wird mittels sogenannter elliptischer Integrale 2. Art angegeben.

Theorie – Prisma und Zylinder

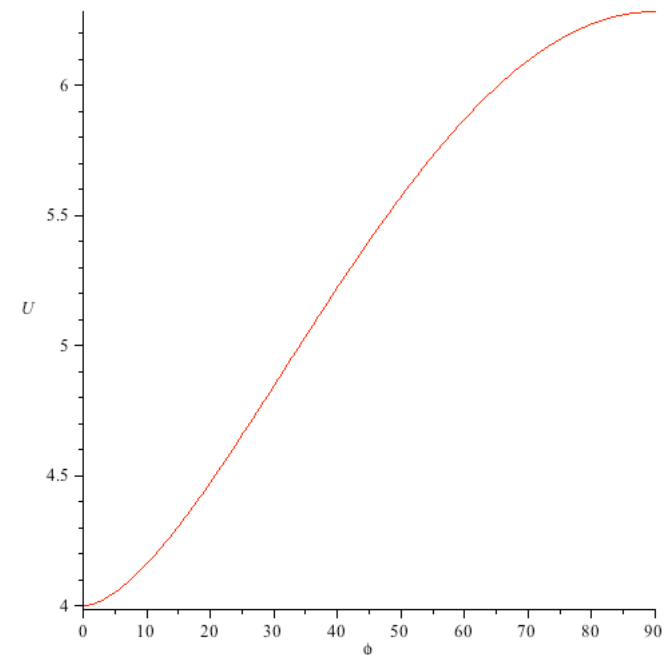
■ Schiefer Zylinder

■ Umfanglänge der Abrollkurve (Ellipse)

$$U = 4 \cdot a \cdot E(b/a) = 4 \cdot r \cdot E(\cos(\varphi)) \text{ mit } 4r \leq U \leq 2\pi r$$

$E(x)$ ist das elliptische Integral zweiter Gattung

```
> U:=(r,phi)->4*r*EllipticE(cos(phi));  
                                U := (r, φ) → 4 r EllipticE(cos(φ))  
[  
> U(r,phi);  
  evalf(%);  
]  
  
4 EllipticE(1/2 √2)  
5.402575524
```



Theorie – Prisma und Zylinder

■ Schiefer Zylinder

■ Mantelfläche

Randkurve ist näherungsweise eine allgemeine Kosinusfunktion mit Amplitude $r \cdot \cos(\varphi)$ und Frequenz $2\pi/U$ (Quotient aus Kreislänge und Periode)

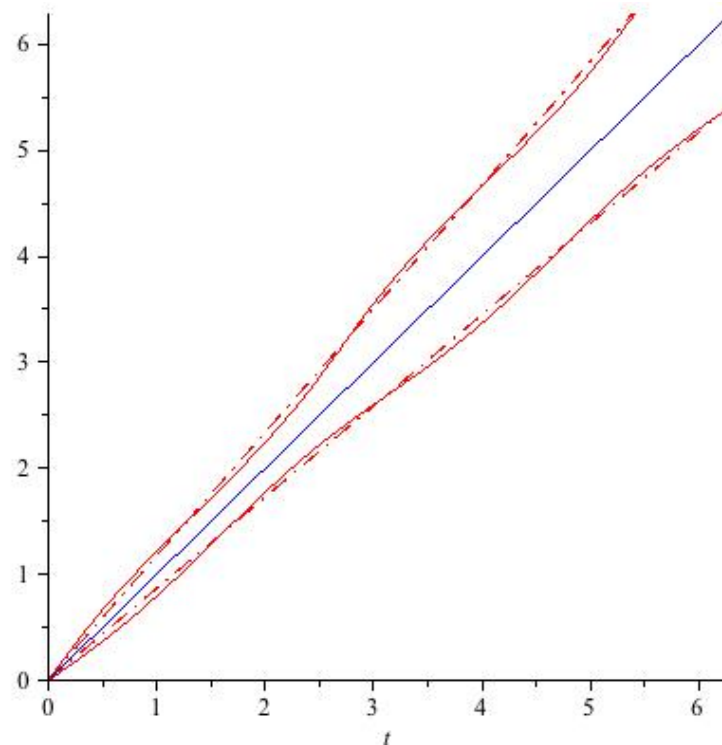
```
> k:=(x,r,phi)->-r*cos(phi)*cos(2*Pi*x/U(r,phi));  
k(x,r,phi);  
evalf(%);
```

$$k := (x, r, \phi) \rightarrow -r \cos(\phi) \cos\left(\frac{2\pi x}{U(r, \phi)}\right)$$
$$-\frac{1}{2}\sqrt{2} \cos\left(\frac{1}{2} \frac{\pi x}{\text{EllipticE}\left(\frac{1}{2}\sqrt{2}\right)}\right)$$
$$-0.7071067810 \cos(1.162998144 x)$$

Theorie – Prisma und Zylinder

■ Schiefer Zylinder

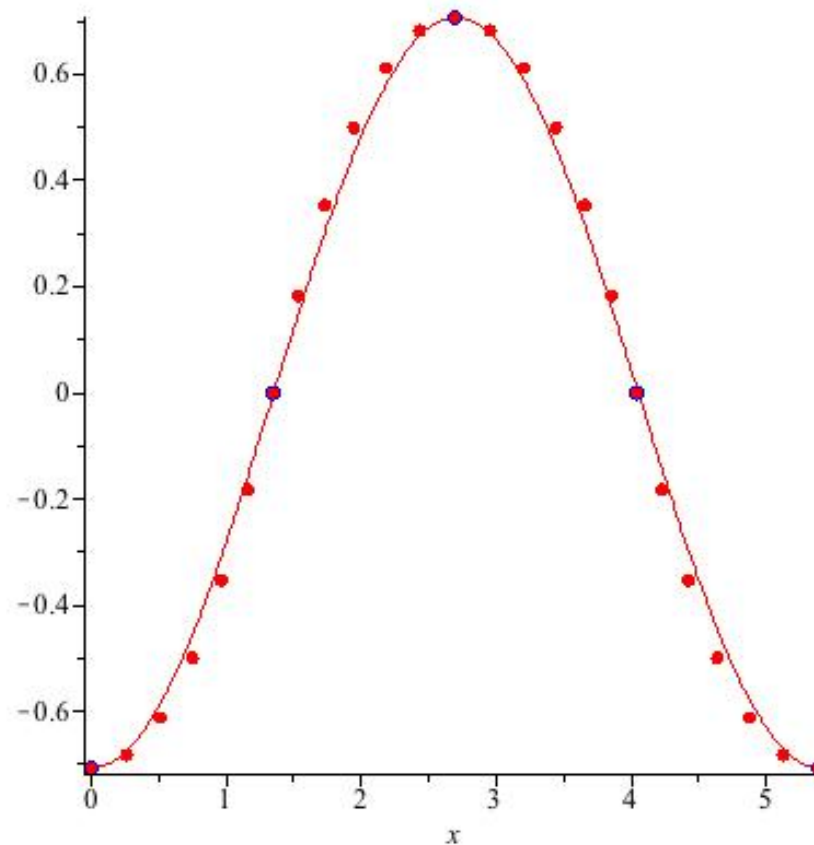
- Abweichungen der Winkelgeschwindigkeiten beim Durchlaufen von Grundkreis (gestrichpunktet) und Abrollellipse (durchgezogen)



Theorie – Prisma und Zylinder

■ Schiefer Zylinder

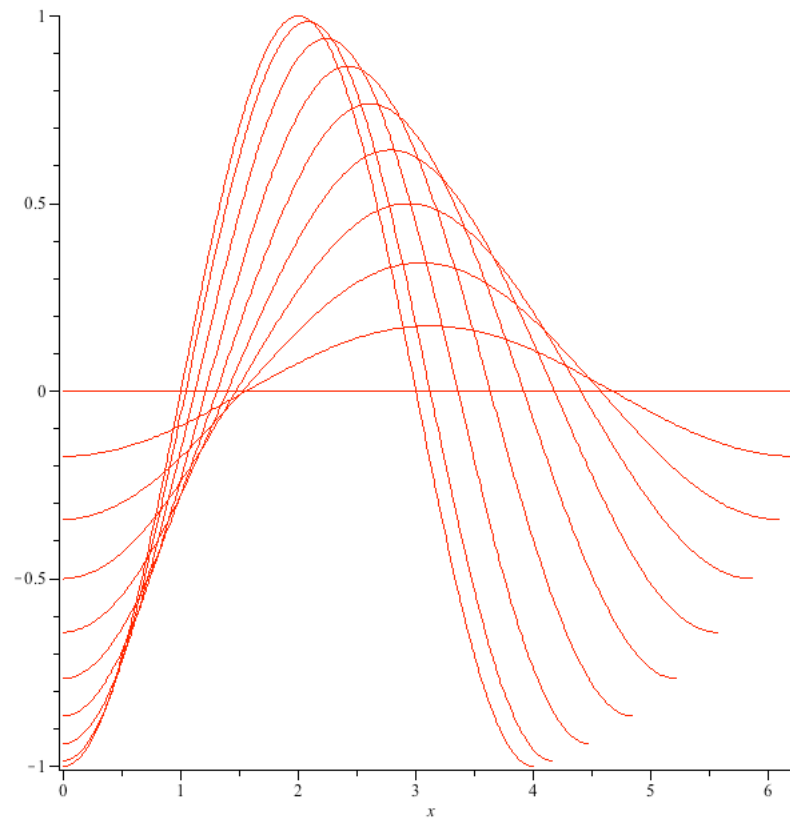
■ Mantelfläche (Näherungsrandkurve) und Abweichung



Theorie – Prisma und Zylinder

■ Schiefer Zylinder

■ Mantelfläche (Näherungsrandkurve) für Schiefen $0^\circ \leq \varphi \leq 90^\circ$

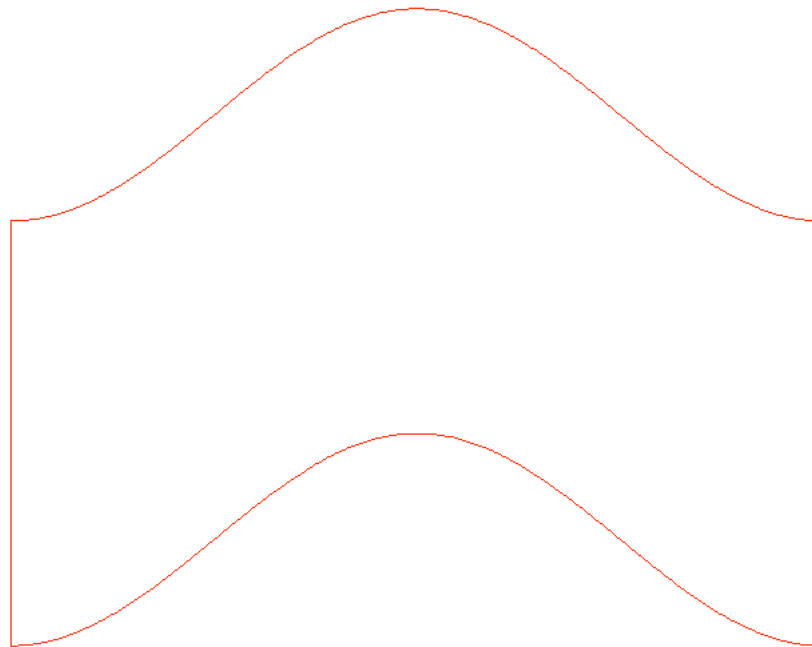


Theorie – Prisma und Zylinder

■ Schiefer Zylinder

■ Mantelfläche

```
> plots[display](  
  plot(k(x,r,phi),x=0..evalf(U(r,phi))),  
  plot(k(x,r,phi)+m,x=0..evalf(U(r,phi))),  
  plot([[0,-r*cos(phi)],[0,m-r*cos(phi)]]),  
  plot([[evalf(U(r,phi)),-r*cos(phi)],[evalf(U(r,phi)),m-r*cos(phi)]]),  
  axes=None,scaling=constrained);
```



Theorie – Pyramide und Kegel

■ Gerade Pyramide

■ Kantenmodell (n = 3)

```
> r:=1;
  h:=2;

                                     r:=1
                                     h:=2

> n:=3;

                                     n:=3

> E:=seq([r*cos(2*i*Pi/n),r*sin(2*i*Pi/n),0],i=0..n-1);
  evalf(E);

                                     E:= $\left[ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 \end{bmatrix} \right]$ 
                                     [[1., 0., 0.], [-0.5000000000, 0.8660254040, 0.], [-0.5000000000, -0.8660254040, 0.]]

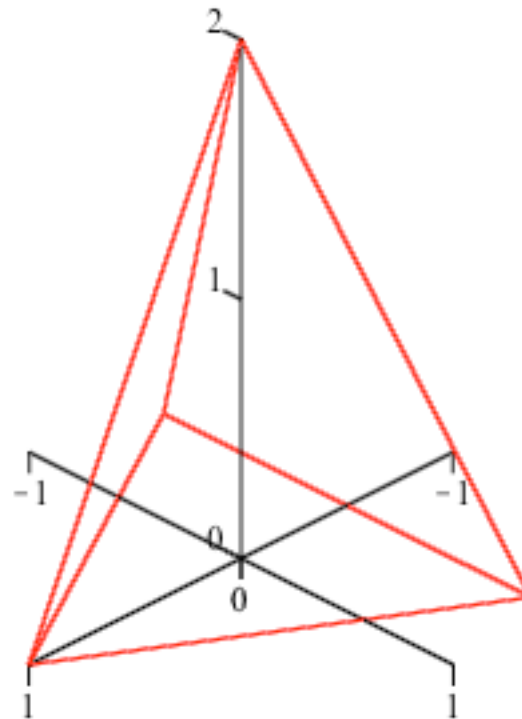
> S:=[0,0,h];
  evalf(S);

                                     S:= $\begin{bmatrix} 0 & 0 & 2 \end{bmatrix}$ 
                                     [0., 0., 2.]

> plots[display](
  plottools[polygon](E,color=red,style=line),
  seq(plottools[line](E[i],S,color=red),i=1..n),
  axes=normal,orientation=[45,60,0],scaling=constrained);
```

Theorie – Pyramide und Kegel

- Gerade Pyramide
 - Kantenmodell ($n = 3$)



Theorie – Pyramide und Kegel

■ Gerade Pyramide

■ Mantelfläche

```
> # Grundkantenlänge
a:=sqrt((E[1][1]-E[2][1])^2+(E[1][2]-E[2][2])^2);
evalf(%);
```

$$a := \sqrt{3}$$
$$1.732050808$$

```
> # Seitenkantenlänge
s:=sqrt(r^2+h^2);
evalf(%);
```

$$s := \sqrt{5}$$
$$2.236067977$$

```
> # Seitenflächenhöhe
hS:=sqrt(s^2-(a/2)^2);
evalf(%);
```

$$hS := \frac{1}{2} \sqrt{17}$$
$$2.061552813$$

Theorie – Pyramide und Kegel

■ Gerade Pyramide

■ Mantelfläche

```
> # Basiswinkel
alpha:=arccos(a/(2*s));
evalf(%);
convert(alpha,degrees):
evalf(%);
```

$$\alpha := \arccos\left(\frac{1}{10} \sqrt{3} \sqrt{5}\right)$$
$$1.173096912$$
$$67.21350201 \text{ degrees}$$

```
> # Winkel an der Spitze
beta:=Pi-2*alpha;
evalf(convert(beta,degrees));
```

$$\beta := \pi - 2 \arccos\left(\frac{1}{10} \sqrt{3} \sqrt{5}\right)$$
$$45.57299598 \text{ degrees}$$

```
> M:= [seq([sum(a*cos(k*beta),k=0..i-1),sum(a*sin(k*beta),k=0..i-1)],i=0..n)]: evalf(%);
      [[0., 0.], [1.732050808, 0.], [2.944486374, 1.236931687], [2.909845360, 2.968636050]]
```

```
> plots[display](
  plottools[polygon]([seq(M[i],i=1..n+1),[a/2,hS]],color=red,style=line),
  seq(plottools[line](M[i],[a/2,hS],color=red,linestyle=3),i=2..n),
  scaling=constrained);
```

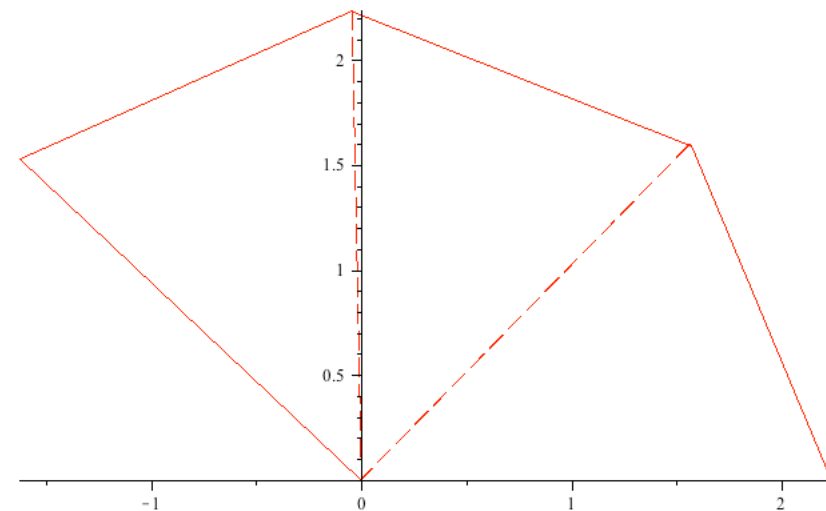
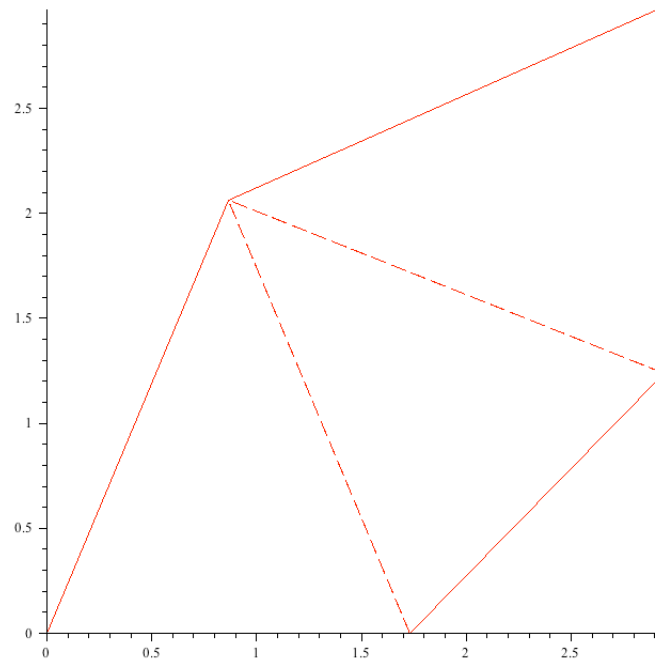
```
> M:= [seq([s*cos(i*beta),s*sin(i*beta)],i=0..n)]:
evalf(%);
[[2.236067977, 0.], [1.565247585, 1.596871941], [-0.04472135688, 2.235620719], [-1.627857485, 1.532997067]]
```

```
> plots[display](
  plottools[polygon]([op(M),[0,0]],color=red,style=line),
  seq(plottools[line](M[i],[0,0],color=red,linestyle=3),i=2..n),
  scaling=constrained);
```

Theorie – Pyramide und Kegel

■ Gerade Pyramide

■ Mantelfläche



Theorie – Pyramide und Kegel

■ Gerader Kegel

■ Parametrisierung

$$K : \vec{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} = \begin{pmatrix} \frac{h-v}{h} \cdot r \cdot \sin(u) \\ \frac{h-v}{h} \cdot r \cdot \cos(u) \\ v \end{pmatrix}$$

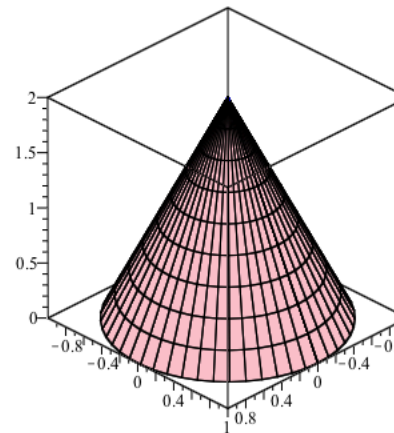
```
> r:=1;  
h:=2;  
  
r:=1  
h:=2  
  
> K:=(u,v)->[(h-v)/h*r*sin(u), (h-v)/h*r*cos(u), v]; K(u,v);  
K := (u, v) -> [ (h-v) r sin(u) / h, (h-v) r cos(u) / h, v ]  
[ 1/2 (2-v) sin(u), 1/2 (2-v) cos(u), v ]
```

Theorie – Pyramide und Kegel

■ Gerader Kegel

■ 3D-Modell

```
> plots[display](  
  plottools[line]([0,0,0],[0,0,h],color=blue,thickness=3),  
  plot3d(K(u,v),u=0..2*Pi,v=0..h,axes=boxed,color=pink,grid=[50,10],orientation=[45,  
  60],scaling=constrained));
```

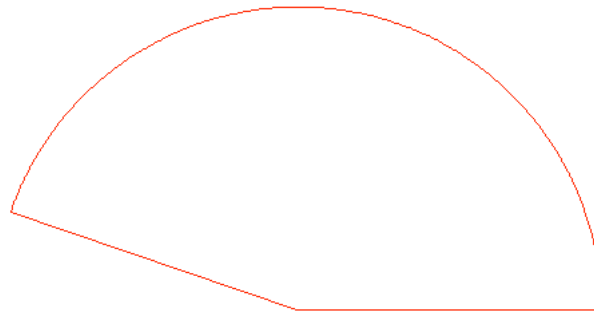


Theorie – Pyramide und Kegel

■ Gerader Kegel

■ Mantelfläche

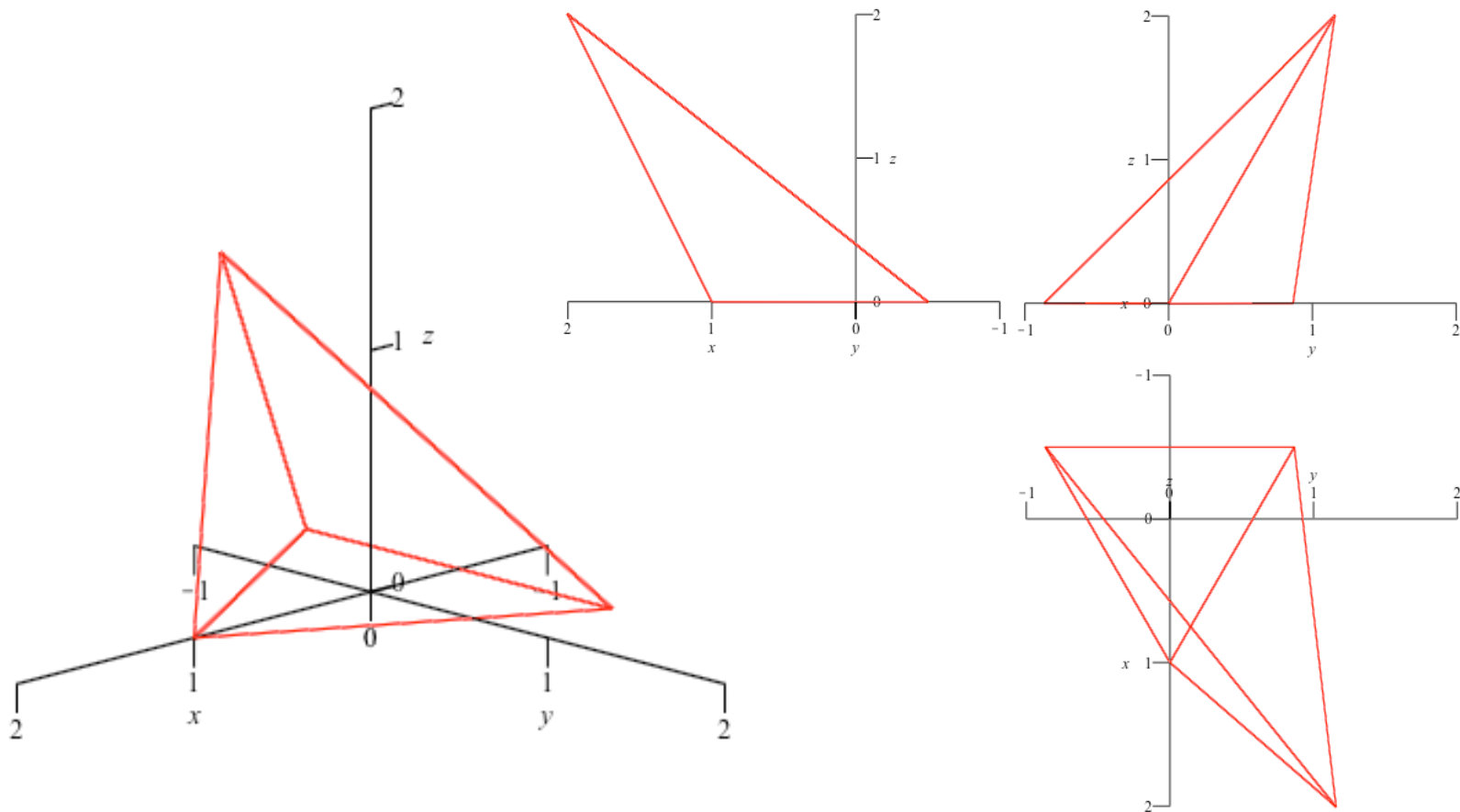
```
> s:=sqrt(r^2+h^2);  
                                      $s := \sqrt{5}$   
                                      $\alpha := \frac{2}{5} \pi \sqrt{5}$   
> alpha:=2*Pi*r/s;  
    evalf(%);  
    evalf(convert(%%,degrees));  
                                     2.809925892  
                                     160.9968943 degrees  
                                      $\alpha := \frac{2}{5} \pi \sqrt{5}$   
> plots[display](  
    plot([s*cos(u),s*sin(u),u=0..alpha]),  
    plot([[0,0],[s,0]]),  
    plot([[0,0],[s*cos(alpha),s*sin(alpha)]]),  
    axes=None,scaling=constrained);
```



Theorie – Pyramide und Kegel

■ Schiefe Pyramide

■ Kantenmodell ($n = 3$, $\varphi = 45^\circ$, $\vartheta = 30^\circ$)



Theorie – Pyramide und Kegel

■ Schiefe Pyramide

■ Mantelfläche

```
> a:=sqrt((E[1][1]-E[2][1])^2+(E[1][2]-E[2][2])^2);  
evalf(%);  
> d:=(X,Y)->sqrt((X[1]-Y[1])^2+(X[2]-Y[2])^2+(X[3]-Y[3])^2);  
> for i from 1 to n do  
  s[i]:=d(E[i],S);  
  evalf(%);  
od;  
> for i from 1 to n-1 do  
  alpha[i]:=arccos(simplify(dotprod(normalize(E[i+1]-E[i]),normalize(S-E[i]))));  
  evalf(convert(alpha[i],degrees));  
od;  
alpha[n]:=arccos(simplify(dotprod(normalize(E[1]-E[n]),normalize(S-E[n]))));  
evalf(convert(alpha[n],degrees));  
> for i from 1 to n-1 do  
  beta[i]:=arccos(simplify(dotprod(normalize(E[i]-E[i+1]),normalize(S-E[i+1]))));  
  evalf(convert(beta[i],degrees));  
od;  
beta[n]:=arccos(simplify(dotprod(normalize(E[n]-E[1]),normalize(S-E[1]))));  
evalf(convert(beta[n],degrees));  
> for i from 1 to n do  
  gamma[i]:=Pi-alpha[i]-beta[i];  
  evalf(convert(gamma[i],degrees));  
od;  
> gamma_M:=sum(gamma['i'], 'i'=1..n);  
evalf(convert(gamma_M,degrees));  
> M:=[[0,0],[a,0]];  
for i from 2 to n do  
  M:=[op(M),[a+a*sum(cos(j*Pi-sum(beta[k],k=1..j)-sum(alpha[k],k=2..j+1)),j=1..i-1),  
a*sum(sin(j*Pi-sum(beta[k],k=1..j)-sum(alpha[k],k=2..j+1)),j=1..i-1)]];  
end;  
evalf(M);  
[[0., 0.], [1.732050808, 0.], [3.171425028, 0.9634323306], [2.226501294, 2.415024263]]
```

Theorie – Pyramide und Kegel

■ Schiefe Pyramide

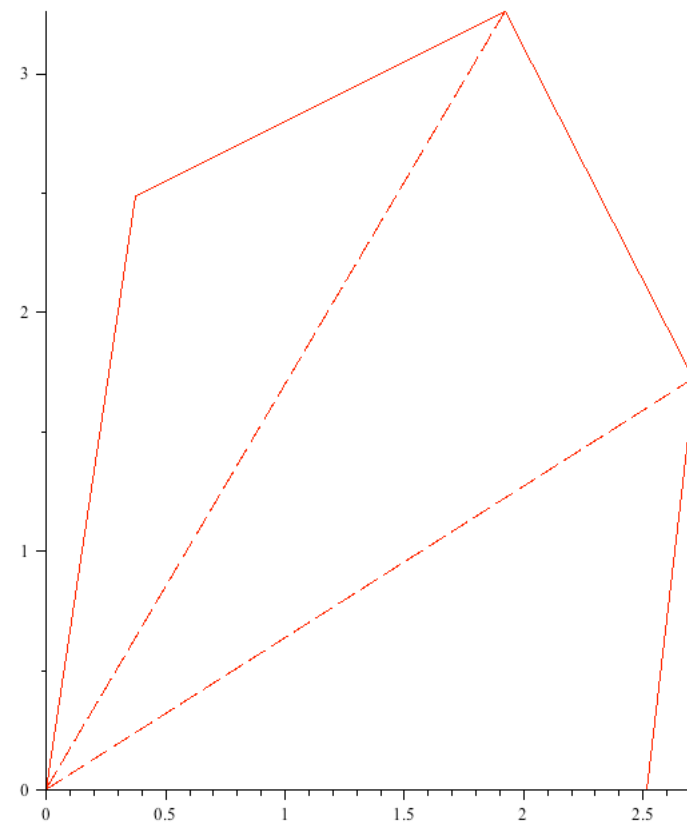
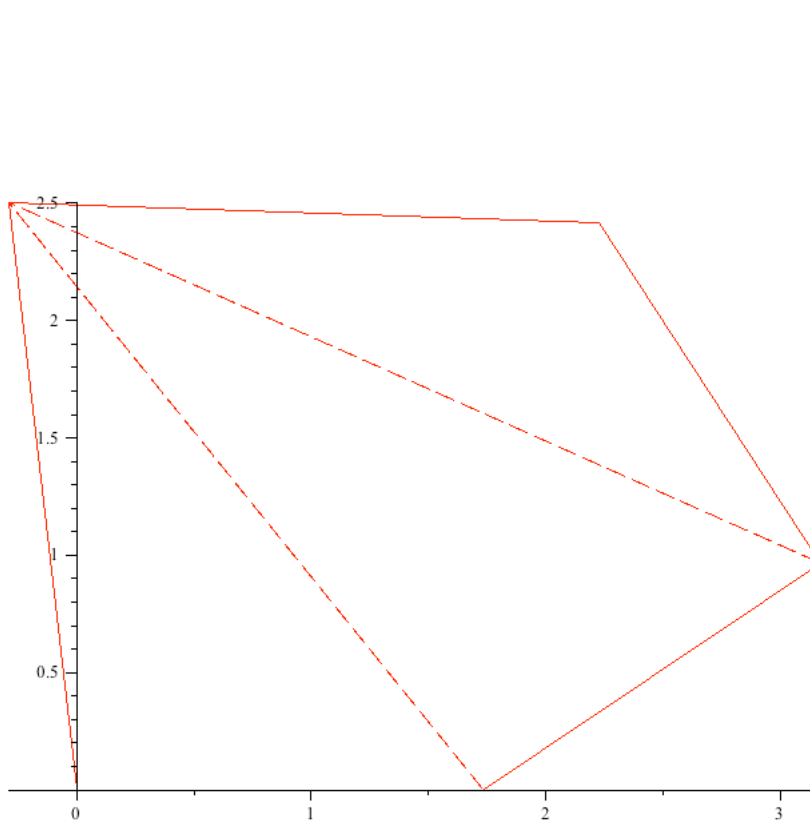
■ Mantelfläche

```
> plots[display](
  plottools[polygon]([op(M), [s[1]*cos(alpha[1]), s[1]*sin(alpha[1])]], color=red, style=
    line),
  seq(plottools[line](M[i], [s[1]*cos(alpha[1]), s[1]*sin(alpha[1])], color=red,
    linestyle=3), i=2..n),
  scaling=constrained);
> M:=[ [s[1], 0], seq([s[i+1]*cos(sum(gamma[k], k=1..i)), s[i+1]*sin(sum(gamma[k], k=1..i))
  ], i=1..n-1), [s[1]*cos(sum(gamma[k], k=1..n)), s[1]*sin(sum(gamma[k], k=1..n))]]:
  evalf(%);
  [[2.516611478, 0.], [2.715291332, 1.720618003], [1.923325907, 3.261004566], [0.3729253623, 2.488827034]]
> plots[display](
  plottools[polygon]([op(M), [0, 0]], color=red, style=line),
  seq(plottools[line](M[i], [0, 0], color=red, linestyle=3), i=2..n),
  scaling=constrained);
```

Theorie – Pyramide und Kegel

■ Schiefe Pyramide

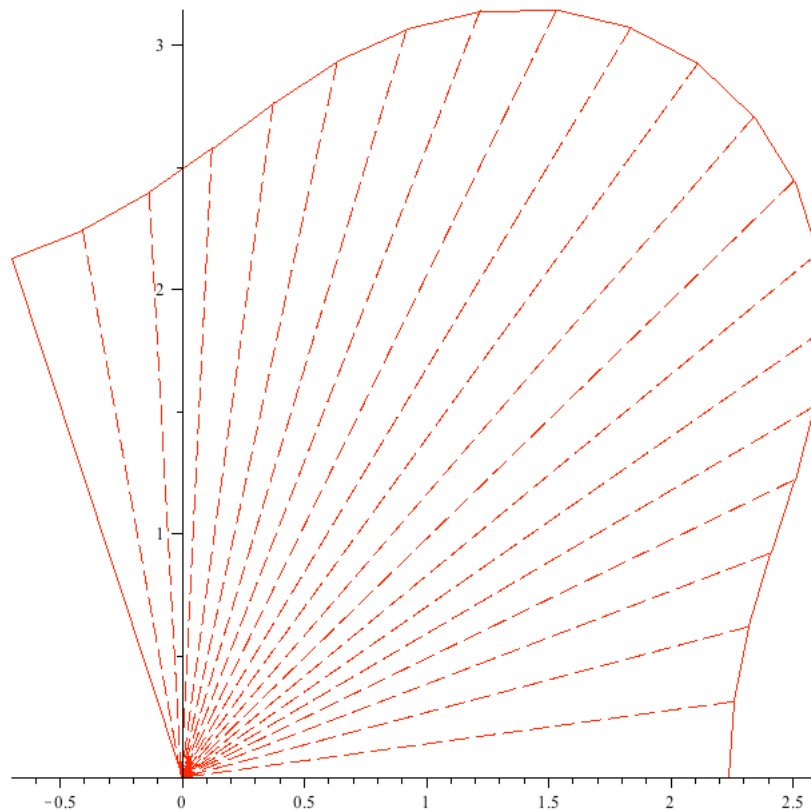
■ Mantelfläche



Theorie – Pyramide und Kegel

■ Schiefer Kegel

■ Näherungslösung (schiefe Pyramide für $n = 20$, $\varphi = 45^\circ$, $\vartheta = 0^\circ$)



Theorie – Pyramide und Kegel

■ Schiefer Kegel

■ Parametrisierung

$$K : \vec{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} = \begin{pmatrix} \frac{h-v}{h} \cdot r \cdot \sin(u) + v \cdot \cot(\varphi) \\ \frac{h-v}{h} \cdot r \cdot \cos(u) \\ v \end{pmatrix}$$

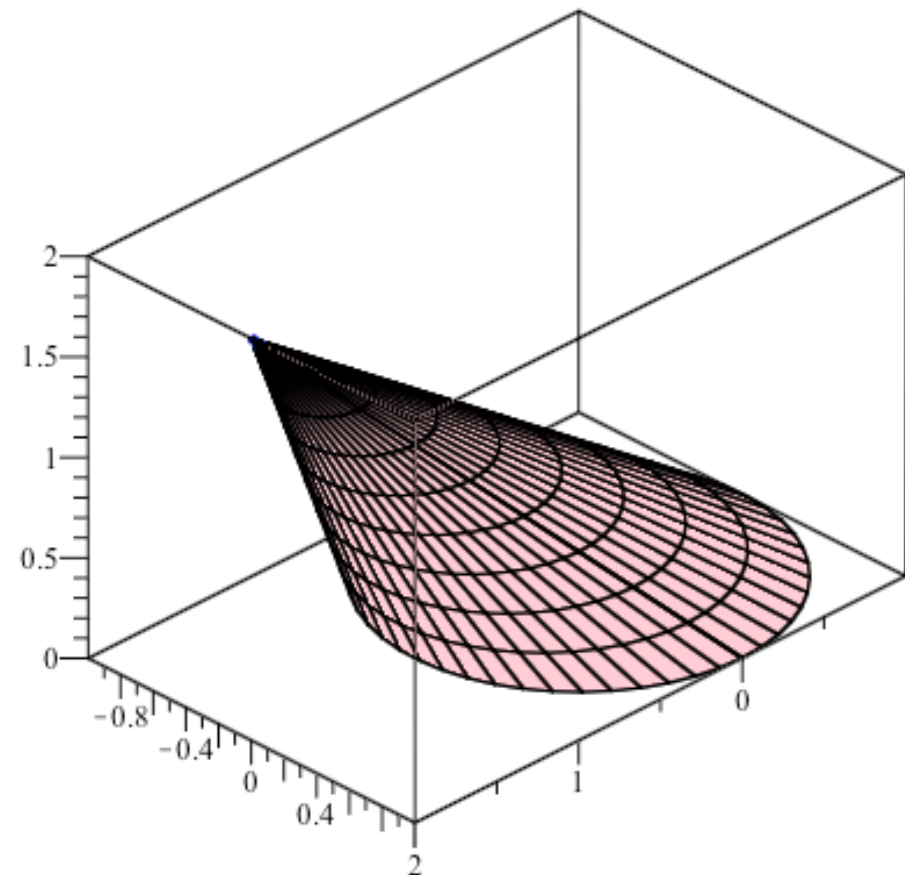
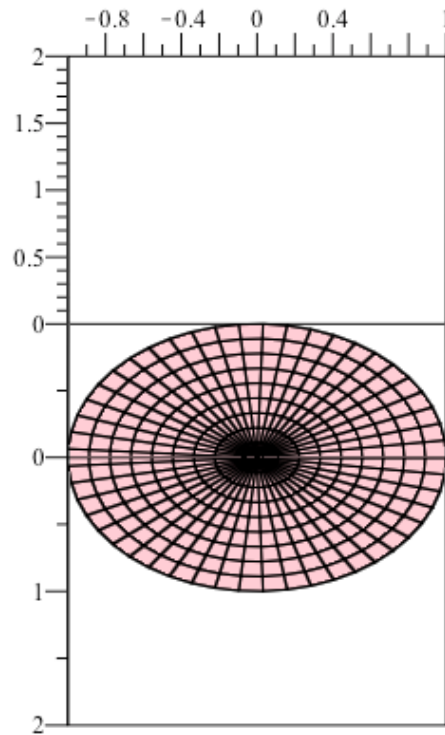
```

> r:=1; h:=2;
                                     r:=1
                                     h:=2
> phi:=45;
                                     φ:=45
> phi:=convert(phi*degrees,radians);
                                     φ:=1/4 π
> K:=(u,v)->[(h-v)/h*r*sin(u)+v*cot(phi), (h-v)/h*r*cos(u), v]; K(u,v);
      K := (u, v) → [ (h-v) r sin(u) / h + v cot(φ), (h-v) r cos(u) / h, v ]
                                     [ 1/2 (2-v) sin(u) + v, 1/2 (2-v) cos(u), v ]
    
```

Theorie – Pyramide und Kegel

■ Schiefer Kegel

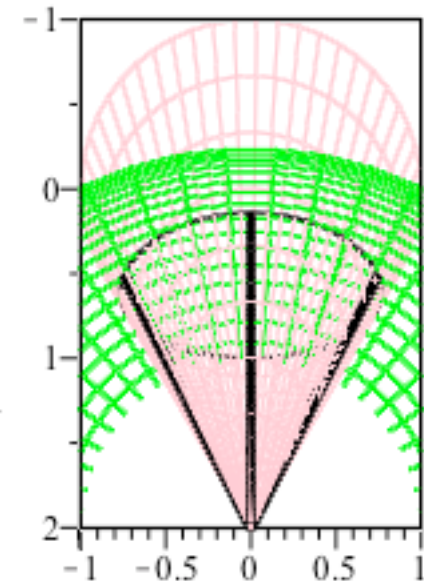
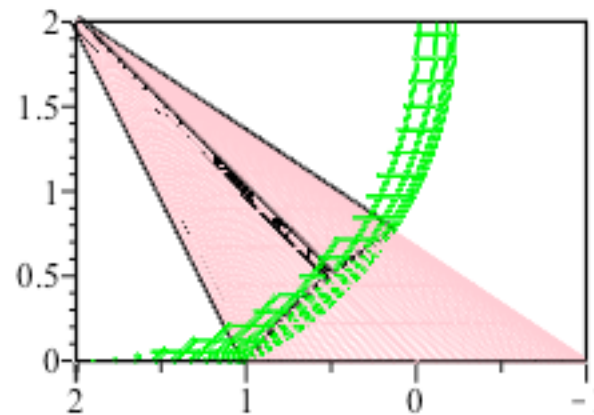
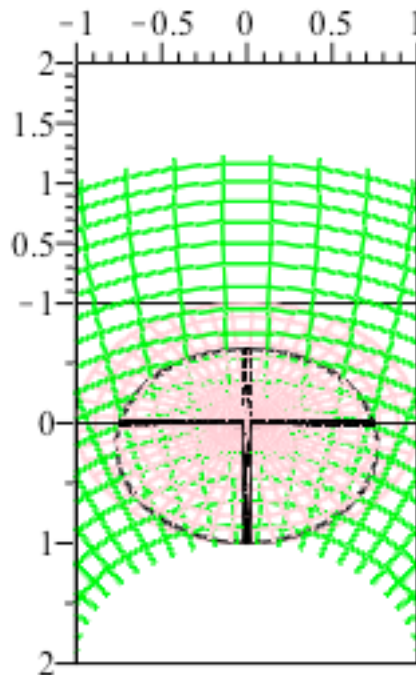
■ 3D-Modell



Theorie – Pyramide und Kegel

■ Schiefer Kegel

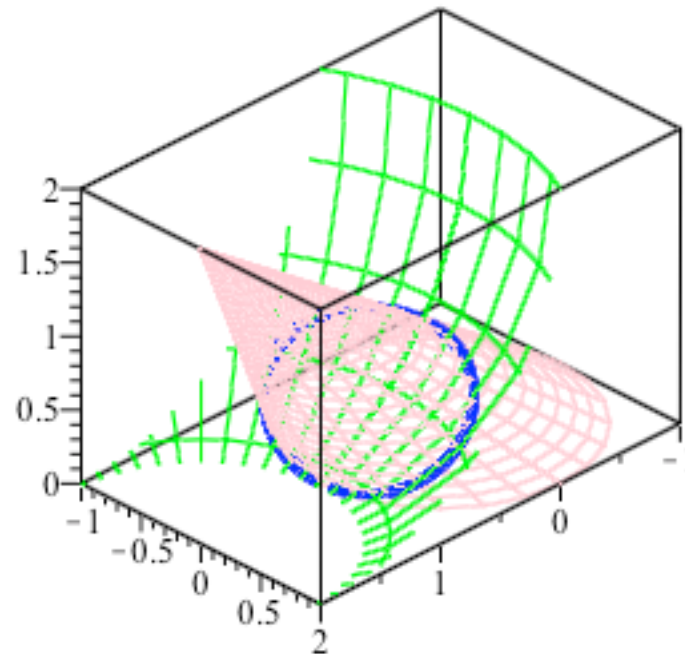
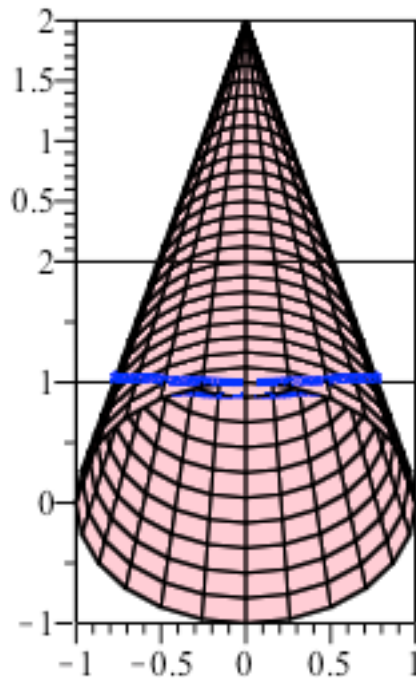
■ Problematik: Abrollkurve



Theorie – Pyramide und Kegel

■ Schiefer Kegel

■ Problematik: Abrollkurve



Theorie – Pyramide und Kegel

■ Schiefer Kegel

■ Mantelfläche (Randkurve) für Schiefen $0^\circ \leq \varphi \leq 90^\circ$

