Vortrag
Karlsruher Institut für Technologie
June 30, 2014, Karlsruhe, Germany

VARIATIONAL SPACE-TIME METHODS FOR THE ELASTIC WAVE EQUATION

M. Bause[1] and U. Köcher[1]

[1] Helmut Schmidt University, University of the Federal Armed Forces Hamburg
Department of Mechanical Engineering · Numerical Methods in Computational Engineering
Holstenhofweg 85, 22043 Hamburg, Germany
{koecher, bause}@hsu-hamburg.de, www.hsu-hamburg.de/mb-mathe

Key words: *Higher order variational space-time FEM, Variational time integration.*

The accurate and reliable numerical approximation of the hyperbolic wave equation is of fundamental importance to the simulation of acoustic, electromagnetic and elastic wave propagation phenomena. Here, we study the elastic wave equation

$$\begin{aligned} \rho(\boldsymbol{x})\,\partial_t \boldsymbol{v}(\boldsymbol{x},t) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{u}(\boldsymbol{x},t)) &= \boldsymbol{f}(\boldsymbol{x},t) \quad \text{in } \Omega\times I\,,\\ \rho(\boldsymbol{x})\,\partial_t \boldsymbol{u}(\boldsymbol{x},t) - \rho(\boldsymbol{x})\,\boldsymbol{v}(\boldsymbol{x},t) &= \boldsymbol{0} \qquad\qquad \text{in } \Omega\times I\,, \end{aligned} \tag{1}$$

written as velocity-displacement formulation and equipped with appropriate initial conditions and boundary conditions, where we denote by $\boldsymbol{v}$ the velocity, by $\boldsymbol{u}$ the displacement, by ρ the mass density, by $\boldsymbol{f}$ the body forces, by $\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon}$ the stress and finally by $\boldsymbol{\varepsilon}$ the strain. Elastic waves appear in the design of integrated structural health monitoring systems for composites. For this, it is strictly necessary to understand phenomenologically and quantitatively wave propagation in layered fibre reinforced composites and the influene of the geometrical and mechanical properties of the system structure. Therefore, the ability to solve numerically the wave equation in three space dimensions is particularly important from the point of view of physical realism.

Recently, variational space-time discretisation schemes were proposed and studied for challenging problems, such as the nonstationary incompressible flow; cf. [3].

In this contribution we will focus on the presentation of variational time integration methods from the variational space-time approach for the hyperbolic elastic wave equation. For the spatial discretisation a symmetric interior penalty discontinuous Galerkin method for anisotropic media is used; cf. [2, 4].

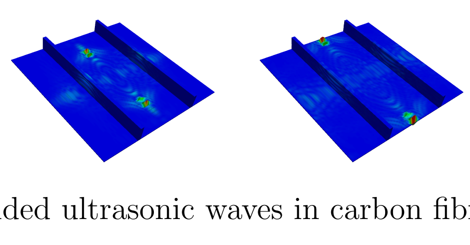

Figure 1: Guided ultrasonic waves in carbon fibre composite.

As common starting point, we choose the weak formulation of Eq. (1), yielding

$$\begin{aligned} \int_I (\rho\, \partial_t \boldsymbol{v}, \boldsymbol{w}) + (\boldsymbol{\sigma}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{w}))\, \mathrm{d}t &= \int_I (\boldsymbol{f}, \boldsymbol{w})\, \mathrm{d}t\,, \quad \forall \boldsymbol{w} \in L^2(I, \boldsymbol{V})\,, \\ \int_I (\rho\, \partial_t \boldsymbol{u}, \boldsymbol{w}) - (\rho\, \boldsymbol{v}, \boldsymbol{w})\, \mathrm{d}t &= 0\,, \qquad\qquad\qquad \forall \boldsymbol{w} \in L^2(I, \boldsymbol{V})\,, \end{aligned} \tag{2}$$

with $\boldsymbol{V} = H_0^1(\partial\Omega_D; \Omega)^3$. We denote by $(\cdot, \cdot)$ the vector-valued L^2 inner product in space. Next, we discretise the time interval I into N disjoint elements $I_n = (t_{n-1}, t_n]$. Finally, we derive variational time integration methods by choosing boundary conditions in time, numerical quadrature in time and the test function space. By choosing a discontinuous test space in time, we can rewrite the resulting finite element in time method as time marching scheme over one or several elements I_n. Doing this, we can easily derive numerous well known schemes, such as the second order in time Crank-Nicolson scheme, which is equivalent to the unconditionally stable second order in time Newmark scheme (cf. [5]), as well as various new higher order continuous and discontinuous Galerkin schemes in time; cf. [1, 2].

From these classes of uniform Galerkin discretisations in space and time an approach of fourth-order accuracy is analysed carefully. More precisely, we use a continuous Petrov-Galerkin method of third order accuracy in time and apply an inexpensive post-processing step, which makes the numerical solution continuously differentiable in time. Further, the efficient solution of the resulting block-matrix system and inherently parallel numerical simulation through domain partioning is adressed. The performance properties of the schemes are illustrated by sophisticated and challenging numerical experiments with complex wave propagation phenomena in heterogeneous and anisotropic media.

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