Multilinear Algebra

1 Tensor Products of Matrices

At first, we will define tensor products for matrices, providing the coordinate version for some of the abstract definitions in the sections of this chapter.

Definition 1.1 The **direct sum** of matrices $A \in \mathrm{Mat}_m$ and $B \in \mathrm{Mat}_n$ is given by

$$A \oplus B = \begin{pmatrix} A & \\ & B \end{pmatrix} \in \mathrm{Mat}_{m+n}.$$

The **tensor product** of two matrices $A \in \mathrm{Mat}_{m,n}$ and $B \in \mathrm{Mat}_{p,q}$ is given by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \in \mathrm{Mat}_{mp,nq},$$

where the a_{ij} are the coefficients of A.

Remark 1.2 For matrices $A, C \in \mathrm{Mat}_m$ and $B, D \in \mathrm{Mat}_n$, we have:

- $(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$.
- $(A \oplus C) \otimes B = (A \otimes B) \oplus (C \otimes B)$.
- $(A \otimes B) = (I_m \otimes B) \cdot (A \otimes I_n) = (A \otimes I_n) \cdot (I_m \otimes B)$.

These identities show that the definition of matrix tensor products is compatible with the abstract definition of tensor products of vector spaces and group representations.

Remark 1.3 For $A \in \mathrm{Mat}_m$, $B \in \mathrm{Mat}_n$ and $v \in \Bbbk^m$, $w \in \Bbbk^n$, it is often convenient to identify $v \otimes w$ with

$$X = v \otimes w^\top = v \cdot w^\top \in \mathrm{Mat}_{m,n},$$

because then the action of $A \otimes B$ on $v \otimes w$ is given by $AXB^\top$.

2 Tensor Algebra

In this section, we give a definition of the tensor algebra $\bigotimes V$ generated by a vector space V. This is the most general associative algebra over V in the sense that it satisfies the relations for associativity, but no other relations. It is constructed as the direct sum of the vector spaces $V^{\otimes k}$ generated by the products $v_1 \otimes \cdots \otimes v_k$ of k elements of V. Proofs for the propositions in this chapter can be found in appendix A of A. Knapp, *Lie Groups beyond an Introduction*.

Definition 2.1 Let V and W be vector spaces over $\Bbbk$. The **tensor product** of V and W is a vector space $V \otimes_{\Bbbk} W$ together with a bilinear map

$$\tau : V \times W \to V \otimes_{\Bbbk} W, \quad (v, w) \mapsto v \otimes w$$

with the following universal property: For every bilinear map $b : V \times W \to U$, where U is some vector space over $\Bbbk$, there exists a unique surjective linear mapping $\varphi : V \otimes_{\Bbbk} W \to U$, such that $\varphi \circ \tau = b$, i.e. the diagramm

$$\begin{array}{ccc} V \times W & \xrightarrow{\tau} & V \otimes_{\Bbbk} W \\ {\scriptstyle b}\downarrow & \swarrow {\scriptstyle \exists_1 \varphi} & \\ U & & \end{array}$$

commutes. When there is no ambiguity about the field $\Bbbk$, we shall write $V \otimes W$ instead of $V \otimes_{\Bbbk} W$.

Remark 2.2 There exists a unique tensor product for V and W. With definition 2.1 one can define tensor products of more than two vector spaces inductively and show that $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$ (see appendix A.1 in A. Knapp, *Lie Groups beyond an Introduction*). Further, the tensor product distributes over direct sums, $V \otimes (W \oplus U) = (V \otimes W) \oplus (V \otimes U)$.

Remark 2.3 For finite-dimensional vector spaces V and W with bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ a basis of $V \otimes W$ is given by

$$\{v_1 \otimes w_1, \ldots, v_1 \otimes w_n, \ \ldots, \ v_m \otimes w_1, \ldots, v_m \otimes w_n\},$$

so we have $\dim(V \otimes W) = mn$. For $V = \Bbbk^m$ and $W = \Bbbk^n$, $v_i \otimes w_j$ coincides with the matrix tensor product.

Definition 2.4 Let V_1, V_2, W_1 and W_2 be vector spaces over $\Bbbk$. For linear mappings $\varphi_1 : V_1 \to W_1$ and $\varphi_2 : V_2 \to W_2$, the **tensor product** of φ_1 and φ_2 is the unique map $\varphi_1 \otimes \varphi_2$ given by the universal property of $W_1 \otimes W_2$ such that

$$b(v_1, v_2) = (\varphi_1 \otimes \varphi_2) \circ \tau,$$

with a bilinear map $b : V_1 \times V_2 \to W_1 \otimes W_2, (v_1, v_2) \mapsto \varphi_1(v_1) \otimes \varphi_2(v_2)$.

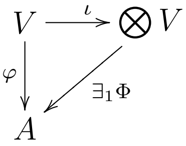
Remark 2.5 With definition 2.4 one can define tensor products for more than two linear mappings inductively.

Remark 2.6 If A is a matrix representation of $\varphi_1 : V_1 \to W_1$, and B is a matrix representation of $\varphi_2 : V_2 \to W_2$, then $A \otimes B$ is a matrix representation of $\varphi_1 \otimes \varphi_2$.

Definition 2.7 Let V be a vector space over $\Bbbk$ and set $V^{\otimes 0} = \Bbbk$. We define the **tensor algebra** generated by V as

$$\bigotimes V = \bigoplus_{k=0}^{\infty} V^{\otimes k}.$$

Proposition 2.8 *The tensor algebra* $\bigotimes V$ *generated by* V *has the following universal property: Let* $\iota : V \to \bigotimes V$ *be the embedding of* V *in* $\bigotimes V$*. If* $\varphi : V \to A$ *is a linear map into an associative algebra* A *with identity, then there exists a unique algebra homomorphism* $\Phi : \bigotimes V \to A$ *with* $\Phi(1) = 1$ *and* $\Phi \circ \iota = \varphi$*, i.e. the diagramm*



commutes.

The tensor algebra is often used to construct associative algebras by taking the quotient over some ideal in $\bigotimes V$ which represents the defining relations of the respective algebra.

3 Symmetric Algebra

In this section we construct a symmetric quotient algebra of $\bigotimes V$ by factoring out the ideal of alternating expressions in $\bigotimes V$. To this end, let

$$\mathfrak{A}^k = \langle v_1 \otimes \cdots \otimes v_k - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \mid v_1, \ldots, v_k \in V, \sigma \in \mathrm{S}_k \rangle \subset V^{\otimes k}$$

be the ideal of alternating expressions in $V^{\otimes k}$.

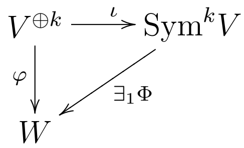
Definition 3.1 The k-fold **symmetric product** of a vector space V is

$$\mathrm{Sym}^k V = V^{\otimes k} / \mathfrak{A}^k.$$

We write $v_1 \cdots v_k$ for the image of $v_1 \otimes \cdots \otimes v_k$ in $\mathrm{Sym}^k V$.

$\mathrm{Sym}^k V$ can be embedded in $V^{\otimes k}$ via the map $v_1 \cdots v_k \mapsto \frac{1}{k!} \sum_{\sigma \in \mathrm{S}_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$.

Proposition 3.2 *The* k*-fold symmetric product* $\mathrm{Sym}^k V$ *has the following universal property: Let* $\iota : V^{\oplus k} \to \mathrm{Sym}^k V$ *be the map* $\iota(v_1, \ldots, v_k) = v_1 \cdots v_k$*. If* $\varphi : V^{\oplus k} \to W$ *is a symmetric* k*-linear map into a vector space* W*, then there exists a unique linear map* $\Phi : \mathrm{Sym}^k V \to W$ *with* $\Phi \circ \iota = \varphi$*, i.e. the diagramm*



commutes.

Remark 3.3 For $V = \Bbbk^n$, the 2-fold symmetric product $\mathrm{Sym}^2 \Bbbk^n$ can be identified with the symmetric matrices via

$$v_1 v_2 \mapsto \frac{1}{2}(v_1 \cdot v_2^\top + v_2 \cdot v_1^\top).$$

Definition 3.4 Let $\mathfrak{A} = \bigoplus_{k=1}^{\infty} \mathfrak{A}^k$ and $\mathrm{Sym}^0 V = \Bbbk$. The **symmetric algebra** over V is defined as

$$\mathrm{Sym} V = (\bigotimes V)/\mathfrak{A} = \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k V.$$

Proposition 3.5 *The symmetric algebra* $\mathrm{Sym} V$ *has the following universal property: Let* $\iota : V \to \mathrm{Sym} V$ *be the embedding of* V *in* $\mathrm{Sym} V$*. If* $\varphi : V \to S$ *is a linear map into an commutative associative algebra* S *with identity, then there exists a unique algebra homomorphism* $\Phi : \mathrm{Sym}^k V \to S$ *with* $\Phi(1) = 1$ *and* $\Phi \circ \iota = \varphi$*, i.e. the diagramm*

$$\begin{array}{ccc} V & \xrightarrow{\ \iota\ } & \mathrm{Sym} V \\ {\scriptstyle \varphi}\downarrow & \swarrow {\scriptstyle \exists_1 \Phi} & \\ S & & \end{array}$$

commutes.

Remark 3.6 For a finite-dimensional vector space V, the elements of a basis $\{v_1, \ldots, v_n\}$ are algebraically independent in $\mathrm{Sym} V$. It follows that $\mathrm{Sym} V$ can be identified with the polynomial ring $\Bbbk[x_1, \ldots, x_n]$. The space $\mathrm{Sym}^k V$ corresponds to the space of homogeneous polynomials of degree k. We have $\dim(\mathrm{Sym}^k V) = \binom{n+k-1}{n-1}$.

Remark 3.7 For the dual V^* of a finite-dimensional vector space V, there is a canonical isomorphism $\mathrm{Sym} V^* \mapsto \Bbbk[x_1, \ldots, x_n]$ given by

$$(v_1^* \cdots v_k^*)(w_1, \ldots, w_k) = \sum_{\sigma \in \mathrm{S}_k} v_1^*(w_{\sigma(1)}) \cdots v_k^*(w_{\sigma(k)})$$

with $v_i^* \in V^*$ and $w_j \in V$.

4 Exterior Algebra

In this section we construct an alternating quotient algebra of $\bigotimes V$ by factoring out the ideal of symmetric expressions in $\bigotimes V$. To this end, let

$$\mathfrak{S}^k = \langle v_1 \otimes \cdots \otimes v_k \mid v_1, \ldots, v_k \in V, \exists\ i \neq j : v_i = v_j \rangle \subset V^{\otimes k}$$

be the ideal of symmetric expressions in $V^{\otimes k}$.

Definition 4.1 The k-fold **exterior product** of a vector space V is

$$\bigwedge^k V = V^{\otimes k}/\mathfrak{S}^k.$$

We write $v_1 \wedge \cdots \wedge v_k$ for the image of $v_1 \otimes \cdots \otimes v_k$ in $\bigwedge^k V$.

$\bigwedge^k V$ can be embedded in $V^{\otimes k}$ via the map $v_1 \wedge \cdots \wedge v_k \mapsto \frac{1}{k!} \sum_{\sigma \in \mathrm{S}_k} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$.

Proposition 4.2 *The k-fold exterior product $\bigwedge^k V$ has the following universal property: Let $\iota : V^{\oplus k} \to \bigwedge^k V$ be the map $\iota(v_1, \dots, v_k) = v_1 \wedge \dots \wedge v_k$. If $\varphi : V^{\oplus k} \to W$ is an alternating k-linear map into a vector space W, then there exists a unique linear map $\Phi : \bigwedge^k V \to W$ with $\Phi \circ \iota = \varphi$, i.e. the diagramm*

$$\begin{array}{ccc} V^{\oplus k} & \xrightarrow{\iota} & \bigwedge^k V \\ {\scriptstyle\varphi}\downarrow & \swarrow {\scriptstyle \exists_1 \Phi} & \\ W & & \end{array}$$

commutes.

Remark 4.3 For $V = \Bbbk^n$, the 2-fold exterior product $\bigwedge^2 \Bbbk^n$ can be identified with the skew-symmetric matrices via

$$v_1 \wedge v_2 \mapsto \frac{1}{2}(v_1 \cdot v_2^\top - v_2 \cdot v_1^\top).$$

Definition 4.4 Let $\mathfrak{S} = \bigoplus_{k=1}^\infty \mathfrak{S}^k$ and $\bigwedge^0 V = \Bbbk$. The **exterior algebra** over V is defined as

$$\bigwedge V = (\bigotimes V)/\mathfrak{S} = \bigoplus_{k=0}^{\infty} \bigwedge^k V.$$

Proposition 4.5 *The exterior algebra $\bigwedge V$ has the following universal property: Let $\iota : V \to \bigwedge V$ be the embedding of V in $\bigwedge V$. If $\varphi : V \to A$ is a linear map into an associative algebra A with identity, then there exists a unique algebra homomorphism $\Phi : \bigwedge^k V \to A$ with $\Phi(1) = 1$ and $\Phi \circ \iota = \varphi$, i.e. the diagramm*

$$\begin{array}{ccc} V & \xrightarrow{\iota} & \bigwedge V \\ {\scriptstyle\varphi}\downarrow & \swarrow {\scriptstyle \exists_1 \Phi} & \\ A & & \end{array}$$

commutes.

Remark 4.6 For an n-dimensional vector space V, we have $\dim(\bigwedge^k V) = \binom{n}{k}$.

Remark 4.7 Let $n = \dim(V)$. Then $\bigwedge^k V^*$ is canonically isomorphic to $(\bigwedge^k V)^*$ by

$$\langle w_1, \dots, w_k | v_1^* \wedge \dots \wedge v_k^* \rangle = \det\big((\langle v_i^* | w_j \rangle)_{ij}\big)$$

with $v_i^* \in V^*$ and $w_j \in V$.

Remark 4.8 Let $n = \dim(V)$ and $b_1, \dots, b_n$ be a basis of V. Then $\bigwedge^{n-k} V$ can be identified with $(\bigwedge^k V)^*$ by

$$\langle v_1 \wedge \dots \wedge v_k | v_{k+1} \wedge \dots \wedge v_n \rangle b_1 \wedge \dots \wedge b_n, = (v_1 \wedge \dots \wedge v_k) \wedge (v_{k+1} \wedge \dots \wedge v_n) \in \bigwedge^n V \cong \Bbbk$$

with $v_i \in V$.