Transformationsgesetze

Gegeben seien eine parametrisierte Fläche

$$x : U \to \mathbb{R}^3,\ (u^1, u^2) \mapsto x(u^1, u^2)$$

sowie eine Umparametrisierung

$$\widetilde{x} : \widetilde{U} \to \mathbb{R}^3,\ (\widetilde{u}^1, \widetilde{u}^2) \mapsto \widetilde{x}(\widetilde{u}^1, \widetilde{u}^2)$$

von x. Insbesondere existieren also differenzierbare Funktionen

$$u^1 = u^1(\widetilde{u}^1, \widetilde{u}^2), \quad u^2 = u^2(\widetilde{u}^1, \widetilde{u}^2) \tag{1}$$

auf $\widetilde{U}$ sodass die Jacobimatrix $J_u(\widetilde{u})$ Rang 2 besitzt,

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$$J_u(\widetilde{u}) = \begin{pmatrix} \frac{\partial u^1}{\partial \widetilde{u}^1} & \frac{\partial u^2}{\partial \widetilde{u}^1} \\ \frac{\partial u^1}{\partial \widetilde{u}^2} & \frac{\partial u^2}{\partial \widetilde{u}^2} \end{pmatrix} .$$

Ist $c(t) = \widetilde{x}(\widetilde{u}(t)) = x(u(\widetilde{u}(t)))$, so erhalten wir durch Ableiten von $u(t) = u(\widetilde{u}(t))$ nach t aus den Gleichungen (1)

$$\dot{u}^1 = \frac{\partial u^1}{\partial \widetilde{u}^1}\dot{\widetilde{u}}^1 + \frac{\partial u^1}{\partial \widetilde{u}^2}\dot{\widetilde{u}}^2, \quad \dot{u}^2 = \frac{\partial u^2}{\partial \widetilde{u}^1}\dot{\widetilde{u}}^1 + \frac{\partial u^2}{\partial \widetilde{u}^2}\dot{\widetilde{u}}^2,$$

in Matrixschreibweise

$$\begin{pmatrix} \dot{u}^1 \\ \dot{u}^2 \end{pmatrix} = J_u(\widetilde{u})^t \begin{pmatrix} \dot{\widetilde{u}}^1 \\ \dot{\widetilde{u}}^2 \end{pmatrix}, \quad (\dot{u}^1, \dot{u}^2) = (\dot{\widetilde{u}}^1, \dot{\widetilde{u}}^2) J_u(\tilde{u}) . \qquad (2)$$

Außerdem

$$\begin{aligned}\tilde{g}_{ij} &= \langle \tilde{x}_{\tilde{u}^i}, \tilde{x}_{\tilde{u}^j} \rangle \\ &= \langle \sum_{k=1}^{2} x_{u^k} \frac{\partial u^k}{\partial \tilde{u}^i}, \sum_{m=1}^{2} x_{u^m} \frac{\partial u^m}{\partial \tilde{u}^j} \rangle \\ &= \sum_{k,m=1}^{2} \frac{\partial u^k}{\partial \tilde{u}^i} \frac{\partial u^m}{\partial \tilde{u}^j} g_{km}, \quad \text{also}\end{aligned}$$

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{pmatrix} = J_u(\tilde{u}) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} J_u(\tilde{u})^t . \qquad (3)$$

Wir haben weiter

$$\begin{aligned}
\sum_{i,j=1}^{2} g_{ij}\dot{u}^i\dot{u}^j &= (\dot{u}^1,\dot{u}^2)\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} \dot{u}^1 \\ \dot{u}^2 \end{pmatrix} \\
&\overset{(2)}{=} (\dot{\tilde{u}}^1,\dot{\tilde{u}}^2)J_u(\tilde{u})\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}J_u(\tilde{u})^t\begin{pmatrix} \dot{\tilde{u}}^1 \\ \dot{\tilde{u}}^2 \end{pmatrix} \\
&\overset{(3)}{=} (\dot{\tilde{u}}^1,\dot{\tilde{u}}^2)\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{pmatrix}\begin{pmatrix} \dot{\tilde{u}}^1 \\ \dot{\tilde{u}}^2 \end{pmatrix} \\
&= \sum_{i,j=1}^{2} \tilde{g}_{ij}\dot{\tilde{u}}^i\dot{\tilde{u}}^j\,.
\end{aligned}$$

Wegen

$$
\begin{aligned}
\det\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{pmatrix} &= \det\left(J_u(\tilde{u})\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} J_u(\tilde{u})^t\right) \\
&= \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \det\left(J_u(\tilde{u})\right) \det\left(J_u(\tilde{u})^t\right) \\
&= \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \det\left(J_u(\tilde{u})\right)^2 \qquad (4)
\end{aligned}
$$

erhalten wir für den Flächeninhalt Folgendes:

Ist $V \subset U$ und $\widetilde{V} = \{(\widetilde{u}^1(v), \widetilde{u}^2(v)) : v \in V\} \subset \widetilde{U}$, so haben wir für $\mathcal{F} = x(V) = \widetilde{x}(\widetilde{V})$ nach der Substitutionsregel für Integrale bei Substitution via (1)

$$\begin{aligned}
\mathcal{O}(\mathcal{F}) &= \iint_V \sqrt{g(u^1, u^2)}du^1 du^2 \\
&= \iint_{\widetilde{V}} \sqrt{g(u^1(\widetilde{u}^1, \widetilde{u}^2), u^2(\widetilde{u}^1, \widetilde{u}^2))} \cdot |\det(J_u(\widetilde{u}))| d\widetilde{u}^1 d\widetilde{u}^2 \\
&\overset{(4)}{=} \iint_{\widetilde{V}} \sqrt{\widetilde{g}(\widetilde{u}^1, \widetilde{u}^2) \det(J_u(\widetilde{u}))^{-2}} \cdot |\det(J_u(\widetilde{u}))| d\widetilde{u}^1 d\widetilde{u}^2 \\
&= \iint_{\widetilde{V}} \sqrt{\widetilde{g}(\widetilde{u}^1, \widetilde{u}^2)} d\widetilde{u}^1 d\widetilde{u}^2 .
\end{aligned}$$