# Differentialgeometrie: Übungsblatt 2

## A1

a) ~~x(t)~~ $x:[0,\infty)\to\mathbb{R}^3$ ; $t\mapsto(1+\cosh t,\ \cos t,\ 1-\sin t)\hat{=}x(t)$

$$x'(t)=(\sinh t,\ -\sin t,\ -\cos t)$$

$$|x'(t)|=\sqrt{\sinh^2 t+\sin^2 t+\cos^2 t}=\sqrt{\sinh^2 t+1}=\sqrt{\cosh^2 t}=\cosh(t)$$

$$s(t)=\int_0^t |x'(t')|\,dt'=\int_0^t \cosh(t')\,dt'=\sinh(t')\Big|_0^t=\sinh(t)$$

$$\Rightarrow t=t(s)=\operatorname{arsinh}(s)=\ln\left(s+\sqrt{s^2+1}\right)$$

Parametrisierung nach Bogenlänge: $\tilde{x}:[0,\infty)\to\mathbb{R}^3$

(beachte: $L(x)=\lim_{t\to\infty}\sinh(t)=\infty$)

$$\tilde{x}(s):=x(t(s))=\left(1+\cosh(\operatorname{arsinh}(s)),\ \cos(\operatorname{arsinh}(s)),\ 1-\sin(\operatorname{arsinh}(s))\right)$$

Es gilt: $|\tilde{x}'(s)|=1$ (Differenziere nach $s$, $\operatorname{arsinh}'(s)=\frac{1}{\sqrt{1+s^2}}$)

b) $x:[0,2]\to\mathbb{R}^3$ ; $t\mapsto\left(\frac{1}{2}t^2,\ 2t,\ \frac{4}{3}t^{3/2}\right)$

$x'(t)=(t,\ 2,\ 2\sqrt{t})$ ; $|x'(t)|=\sqrt{t^2+4+4t}=\sqrt{\underbrace{(t+2)^2}_{\geq 0}}=t+2$

$$s(t)=\int_0^t (t'+2)\,dt'=\frac{1}{2}t^2+2t$$

$$\Leftrightarrow\ t^2+4t=2s\ \Leftrightarrow\ (t^2+4t+4)-4=2s$$

$$\Leftrightarrow\ (t+2)^2=2s+4\ \Leftrightarrow\ t+2=\pm\sqrt{2s+4}$$

$$\Leftrightarrow\ t=-2\pm\sqrt{2s+4}\quad(*)$$

Wegen $s\geq 0$ auf $[0,2]$ und $t\in[0,2]$ folgt aus $(*)$:

$t=t(s)=-2+\sqrt{2s+4}$, denn $\tilde{t}:=-2-\underbrace{\sqrt{2s+4}}_{\geq 2}<0$

Parametrisierung nach Bogenlänge: $L(x) = s(2) = 6$

$$\tilde{x} : [0,6] \to \mathbb{R}^3$$

$$\tilde{x}(s) := x(t(s)) = \left( \frac{1}{2}\left[4 + 2s + 4 - 4\sqrt{2s+4}\right], -4 + 2\sqrt{2s+4}, \frac{4}{3}\left(\sqrt{2s+4} - 2\right)^{3/2} \right)$$

$$= \left( 4 + s - 2\sqrt{2s+4}, \; 2\sqrt{2s+4} - 4, \; \frac{4}{3}\left(\sqrt{2s+4} - 2\right)^{3/2} \right)$$

Es gilt $|\tilde{x}'(s)| = 1$. (Differenzieren nach s, Kettenregel)

## A2) Kettenlinie und Traktrix:

a) Kettenlinie: $x : [-8,8] \to \mathbb{R}^3$, $t \mapsto (t, \cosh t, 0) \equiv x(t)$

$x'(t) = (1, \sinh t, 0) \neq (0,0,0)$ für alle $t \in [-8,8]$, d.h. regulär.

$|x'(t)| = \sqrt{1 + \sinh^2 t} = \cosh(t)$

$$\Rightarrow L(x) = \int_{-8}^{8} \cosh(t)\, dt = \sinh(t)\Big|_{-8}^{8} = \sinh(8) - \sinh(-8)$$

$$= 2\sinh(8)$$

Tangenteinheitsvektor: $T(t) = \dfrac{x'(t)}{|x'(t)|} = \left( \dfrac{1}{\cosh(t)}, \tanh(t), 0 \right)$

Kurz: $T(t) = \left( \dfrac{1}{\cosh t}, \tanh(t), 0 \right)$

b) Traktrix:

$x : (0,8) \to \mathbb{R}^3 \; ; \; t \mapsto \left(t - \tanh t, \frac{1}{\cosh t}, 0\right)$

$$x'(t) = \left(1 - \left[\frac{\sinh^2(t)}{\cosh^2(t)} + \frac{\cosh(t)}{\cosh(t)}\right], -\frac{\sinh(t)}{\cosh^2(t)}, 0\right)$$

$$= \left(\tanh^2(t), -\frac{\sinh(t)}{\cosh^2(t)}, 0\right) \neq (0,0,0) \quad \text{auf } (0,8)$$

da $\sinh(t) = 0$ nur für $t = 0$., d.h. $x$ ist regulär.

$$|x'(t)| = \frac{\sinh(t)}{\cosh^2(t)} \sqrt{\sinh^2(t) + 1} = \frac{\sinh(t)}{\cosh(t)} = \tanh(t)$$

$$\Rightarrow L(x) = \int_0^8 \tanh(t')\, dt' = \ln \cosh(t') \Big|_0^8 = \ln(\cosh(8))$$

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A3

Zykloide

$x : [0, 2\pi] \to \mathbb{R}^3, \; t \mapsto r(t - \sin t, 1 - \cos t, 0)$

$x'(t) = r(1 - \cos t, \sin t, 0).$

$|x'(t)| = r\sqrt{(1-\cos t)^2 + \sin^2 t} = r\sqrt{2 - 2\cos t} = r\sqrt{4\sin^2\frac{t}{2}} = 2r\sin\frac{t}{2}$

$$L(x) = \int_0^{2\pi} 2r\sin\frac{t}{2}\, dt \underset{dt = 2du}{\overset{t = 2u}{=}} 2r\int_0^{\pi} \sin u \, 2du = 4r\underbrace{(-\cos u)\Big|_0^{\pi}}_{=2} = 8r$$

$$T(t) = \frac{x'(t)}{|x'(t)|} = \frac{1}{2\sin\frac{t}{2}}(1 - \cos t, \sin t, 0)$$

$x''(t) = r(\sin t, \cos t, 0)$

$x'(t) \times x''(t) = r^2(0, 0, (1-\cos t)\cos t - \sin^2 t) = r^2(0, 0, \cos t - 1)$

$$\kappa(t) = \frac{|x'(t) \times x''(t)|}{|x'(t)|^3} = \frac{\cos t - 1}{8r\sin^3 t/2} = \frac{-2\sin^2\frac{t}{2}}{8r\sin^3\frac{t}{2}} = \frac{-1}{4r\sin\frac{t}{2}}$$

<u>Beachte:</u> $\kappa(\pi) = \frac{1}{4r}$, $\kappa(t) \to \infty$ für $t \to 0$, $t \to 2\pi$

Alternative Parameterdarstellung:

$$x : [0, 2\pi] \to \mathbb{R}^3, \quad t \mapsto (1+t-\sin t-\cos t,\ 1-t+\sin t-\cos t,\ 0)$$

$$x'(t) = (1-\cos t+\sin t,\ -1+\cos t+\sin t,\ 0)$$

$$|x'(t)|^2 = (\sin t + 1 - \cos t)^2 + (\sin t - (1-\cos t))^2$$

$$= 2\sin^2 t + 2(1-\cos t)^2 = \underline{2\sin^2 t} + 2 - 4\cos t + \underline{2\cos^2 t}$$

$$= 4 - 4\cos t = 8\sin^2\frac{t}{2}, \quad \text{also } |x'(t)| = 2\sqrt{2}\sin\frac{t}{2}$$

$$L(x) = \int_0^{2\pi} 2\sqrt{2}\sin\frac{t}{2}\,dt \underset{dt=2du}{\overset{t=2u}{=}} 2\sqrt{2}\int_0^{\pi}\sin u\cdot 2\,du = 4\sqrt{2}\,\underbrace{(-\cos u)\Big|_0^{\pi}}_{=2} = 8\sqrt{2}$$

$$T(t) = \frac{x'(t)}{|x'(t)|} = \frac{1}{2\sqrt{2}\sin\frac{t}{2}}(1-\cos t+\sin t,\ -1+\cos t+\sin t)$$

$$x''(t) = (\sin t+\cos t,\ -\sin t+\cos t,\ 0)$$

$$x'(t)\times x''(t) = \big(0,\ 0,\ (1-\cos t+\sin t)(-\sin t+\cos t)$$
$$-(-1+\cos t+\sin t)(\sin t+\cos t)\big)$$

$$|x'(t)\times x''(t)| = -\cancel{\sin t} + \underline{\cos t} + \cancel{\cos t\sin t} - \cos^2 t - \sin^2 t - \underline{\sin t\cos t}$$
$$+\cancel{\sin t} + \underline{\cos t} - \cancel{\cos t\sin t} - \cos^2 t - \sin^2 t - \sin t\,\underline{\cos t}$$

$$= 2\cos t - 2 = -4\sin^2\frac{t}{2}$$

$$\kappa(t) = \frac{|x'(t)\times x''(t)|}{|x'(t)|^3} = \frac{-4\sin^2\frac{t}{2}}{8\sqrt{8}\sin^3\frac{t}{2}} = \frac{-1}{4\sqrt{2}\sin\frac{t}{2}}$$