Aufgabe 4

$$x(u^1,u^2) = (u^1, u^2, u^1 u^2)$$

$$x_{u^1} = (1, 0, u^2) \qquad g_{11} = 1 + (u^2)^2, \quad g_{12} = u^1 u^2 = g_{21}$$

$$x_{u^2} = (0, 1, u^1) \qquad g_{22} = 1 + (u^1)^2$$

$$x_{u^1} \times x_{u^2} = (-u^2, -u^1, 1)$$

$$|x_{u^1} \times x_{u^2}|^2 = (u^2)^2 + (u^1)^2 + 1 = g$$

$$n = \frac{1}{\sqrt{1 + (u^1)^2 + (u^2)^2}} (-u^2, -u^1, 1) = \frac{1}{\sqrt{g}} (-u^2, -u^1, 1)$$

$$x_{u^1 u^1} = (0, 0, 0) = x_{u^2 u^2} \quad \rightsquigarrow \quad b_{11} = b_{22} = 0$$

$$x_{u^1 u^2} = (0, 0, 1) = x_{u^2 u^1} \quad \rightsquigarrow \quad b_{12} = \frac{1}{\sqrt{g}} = b_{21}$$

$$b = 0 - \frac{1}{\sqrt{g}} \cdot \frac{1}{\sqrt{g}} = -\frac{1}{g}$$

$$K = \frac{b}{g} = -\frac{1}{g^2} = \frac{-1}{(1 + (u^1)^2 + (u^2)^2)^2}$$

$$H = \frac{1}{2g} (\underbrace{b_{11} g_{22}}_{=0} - 2 g_{12} b_{12} + \underbrace{b_{22} g_{11}}_{=0}) = -\frac{g_{12} b_{12}}{g}$$

$$= -\frac{1}{g} \cdot u^1 u^2 \cdot \frac{1}{\sqrt{g}} = \frac{-u^1 u^2}{g^{3/2}} = \frac{-u^1 u^2}{(1 + (u^1)^2 + (u^2)^2)^{3/2}}$$

$$\tilde{x}(v^1,v^2) = (v^1+v^2,\ v^1-v^2,\ (v^1)^2-(v^2)^2)$$

$$\tilde{x}_{v^1} = (1,1,2v^1) \qquad \tilde{g}_{11} = 2+4(v^1)^2, \quad \tilde{g}_{12} = -4v^1v^2 = \tilde{g}_{21}$$

$$\tilde{x}_{v^2} = (1,-1,-2v^2) \qquad \tilde{g}_{22} = 2+4(v^2)^2$$

$$\tilde{x}_{v^1} \times \tilde{x}_{v^2} = (-2v^2+2v^1,\ 2v^1+2v^2,\ -2)$$

$$= 2(v^1-v^2,\ v^1+v^2,\ -1)$$

$$|\tilde{x}_{v^1} \times \tilde{x}_{v^2}|^2 = 4\Big(\underbrace{(v^1-v^2)^2+(v^1+v^2)^2}_{=2(v^1)^2+2(v^2)^2}+1\Big) = \tilde{g}$$

$$\rightsquigarrow \quad \tilde{n} = \frac{1}{\sqrt{1+2(v^1)^2+2(v^2)^2}}(v^1-v^2,\ v^1+v^2,\ -1)$$

$$\tilde{x}_{v^1v^1} = (0,0,2) \qquad \tilde{b}_{11} = \frac{-2}{\sqrt{1+2(v^1)^2+2(v^2)^2}}$$

$$\tilde{x}_{v^1v^2} = (0,0,0) = \tilde{x}_{v^2v^1} \qquad \tilde{b}_{12} = 0 = \tilde{b}_{21}$$

$$\tilde{x}_{v^2v^2} = (0,0,-2) \qquad \tilde{b}_{22} = \frac{2}{\sqrt{1+2(v^1)^2+2(v^2)^2}}$$

$$\tilde{b} = \frac{-4}{1+2(v^1)^2+2(v^2)^2}$$

$$\tilde{K} = \frac{\tilde{b}}{\tilde{g}} = \frac{-4}{4(1+2(v^1)^2+2(v^2)^2)^2} = \frac{-1}{(1+2(v^1)^2+2(v^2)^2)^2}$$

$$\tilde{H} = \frac{1}{2\tilde{g}}\Big(\tilde{b}_{11}\tilde{g}_{22} - \underbrace{2\tilde{g}_{12}\tilde{b}_{12}}_{=0} + \tilde{b}_{22}\tilde{g}_{11}\Big)$$

$$= \frac{1}{2\tilde{g}}\left(\frac{-2(2+4(v^2)^2}{\sqrt{1+2(v^1)^2+2(v^2)^2}} + \frac{2(2+4(v^1)^2)}{\sqrt{1+2(v^1)^2+2(v^2)^2}}\right)$$

$$= \frac{1}{\tilde{g}^{3/2}}\left(-4-8(v^2)^2+4+8(v^1)^2\right) = \frac{8}{\tilde{g}^{3/2}}((v^1)^2-(v^2)^2)$$

$$= \frac{(v^1)^2-(v^2)^2}{(1+2(v^1)^2+2(v^2)^2)^{3/2}}$$

Aufgabe 5

$x(u^1,u^2) = (u^1, u^2, e^{u^1} - e^{u^2})$

$x_{u^1} = (1, 0, e^{u^1})$, $\quad g_{11} = 1 + e^{2u^1}$, $\quad g_{12} = e^{u^1} e^{u^2} = g_{21}$

$x_{u^2} = (0, 1, e^{u^2})$ $\quad g_{22} = 1 + e^{2u^2}$

$x_{u^1} \times x_{u^2} = (-e^{u^1}, -e^{u^2}, 1)$

$g = |x_{u^1} \times x_{u^2}|^2 = e^{2u^1} + e^{2u^2} + 1$

$$n = \frac{1}{\sqrt{e^{2u^1} + e^{2u^2} + 1}} (-e^{u^1}, -e^{u^2}, 1)$$

$x_{u^1 u^1} = (0, 0, e^{u^1})$ $\quad b_{11} = \frac{e^{u^1}}{\sqrt{e^{2u^1} + e^{2u^2} + 1}}$

$x_{u^1 u^2} = x_{u^2 u^1} = 0$ $\quad b_{12} = b_{21} = 0$

$x_{u^2 u^2} = (0, 0, e^{u^2})$ $\quad b_{22} = \frac{e^{u^2}}{\sqrt{e^{2u^1} + e^{2u^2} + 1}}$

$$b = \frac{e^{u^1} e^{u^2}}{e^{2u^1} + e^{2u^2} + 1}$$

$$K = \frac{b}{g} = \frac{e^{u^1} e^{u^2}}{(e^{2u^1} + e^{2u^2} + 1)^2}$$

$K > 0$ für alle $(u^1, u^2) \in \mathbb{R}^2 \Rightarrow$ Fläche besteht nur elliptische Punkte

$$H = \frac{1}{2g}(g_{11} b_{22} - 2 g_{12} b_{12} + g_{22} b_{11})$$

$$= \frac{1}{2(e^{2u^1} + e^{2u^2} + 1)^{3/2}} \left((1 + e^{2u^1}) e^{u^2} + (1 + e^{2u^2}) e^{u^1} \right)$$

$$K_{1,2} = H \pm \sqrt{H^2 - K}$$