A3 Christoffelsymbole von Graphen

Sei $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ eine differenzierbare Funktion.
Wir bestimmen die Christoffelsymbole der parametrisierten Flächen:

$x: U \to \mathbb{R}^3 \quad ; \quad (u_1, u_2) \mapsto (u_1, u_2, f(u_1, u_2))$

Beispiele: siehe A1 a), b) Blatt 13: Elliptisches u. hyperbolisches Paraboloid.

$x_{u_1} = (1, 0, f_{u_1}) \quad ; \quad x_{u_2} = (0, 1, f_{u_2})$

$g_{11} = 1 + f_{u_1}^2 \quad ; \quad g_{22} = 1 + f_{u_2}^2 \quad ; \quad g_{12} = g_{21} = f_{u_1} f_{u_2}$

$\Rightarrow \quad g = 1 + f_{u_1}^2 + f_{u_2}^2$

und:

$$g_{11,1} = 2 f_{u_1} f_{u_1 u_1} \quad ; \quad g_{11,2} = 2 f_{u_1} f_{u_1 u_2}$$

$$g_{12,1} = g_{21,1} = f_{u_1 u_1} f_{u_2} + f_{u_1} f_{u_1 u_2}$$

$$g_{12,2} = g_{21,2} = f_{u_1 u_2} f_{u_2} + f_{u_1} f_{u_2 u_2}$$

$$g_{22,1} = 2 f_{u_2} f_{u_2 u_1} \quad ; \quad g_{22,2} = 2 f_{u_2} f_{u_2 u_2}$$

Sei $(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$. Dann ist die Inverse gegeben durch:

$$(g_{ij})^{-1} = (g^{lk}) = \frac{1}{g} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}$$ (Überprüfen!). Also gilt:

$$g^{11} = \frac{g_{22}}{g} = \frac{1 + f_{u_2}^2}{g} \quad ; \quad g^{22} = \frac{g_{11}}{g} = \frac{1 + f_{u_1}^2}{g} \quad ; \quad g^{12} = g^{21} = -\frac{g_{12}}{g} = \frac{-f_{u_1} f_{u_2}}{g}$$

Die Christoffelsymbole sind gegeben durch:

$$\Gamma^l_{ik} := \frac{1}{2} g^{lj} \left(g_{ji,k} + g_{jk,i} - g_{ik,j} \right)$$; (über den Index j wird summiert, $j = 1, 2$)

Wegen $g_{mn} = g_{nm}$ gilt: $\Gamma^l_{ik} = \Gamma^l_{ki}$

II

$$\Gamma^1_{11} = \underbrace{\frac{1}{2}\, g^{11}\left(g_{11,1} + \overbrace{g_{11,1} - g_{11,1}}^{=0}\right)}_{\delta=1-\text{Term}} + \underbrace{\frac{1}{2}\, g^{12}\left(\overbrace{g_{12,1} + g_{21,1}}^{=2g_{12,1}} - g_{11,2}\right)}_{\delta=2-\text{Term}}$$

$$= \frac{1}{2}\,\frac{(1+f_{u_2}^2)}{g}\cdot 2 f_{u_1} f_{u_1u_1} + \frac{1}{2}\,\frac{(-f_{u_1}f_{u_2})}{g}\left[2\left(f_{u_1u_1} f_{u_2} + f_{u_1} f_{u_2u_1}\right) - 2 f_{u_1} f_{u_1u_2}\right]$$

$$= \frac{1}{g}\left[f_{u_1} f_{u_1u_1} + \cancel{f_{u_1} f_{u_2}^2 f_{u_1u_1}} - \cancel{f_{u_1} f_{u_2}^2 f_{u_1u_1}}\,\cancel{2} - f_{u_1}^2 \cancel{f_{u_2} f_{u_2u_1}} + \cancel{2 f_{u_1}^2 f_{u_2} f_{u_1u_2}}\right]$$

$$= \frac{f_{u_1} f_{u_1u_1}}{g} = \frac{f_{u_1} f_{u_1u_1}}{1+f_{u_1}^2+f_{u_2}^2}$$

$$\Gamma^2_{11} = \underbrace{\frac{1}{2}\, g^{21}\left(g_{11,1}\right)}_{\delta=1-\text{Term}} + \underbrace{\frac{1}{2}\, g^{22}\left(2\, g_{12,1} - g_{11,2}\right)}_{\delta=2-\text{Term}}$$

$$= \frac{1}{g}\left[-f_{u_1}^2 f_{u_2} f_{u_1u_1} + (1+f_{u_1}^2)\left(2\, f_{u_1u_1} f_{u_2} + \cancel{f_{u_1} f_{u_2u_1}} - \cancel{f_{u_1} f_{u_1u_2}}\right)\right]$$

$$= \frac{1}{g}\left[f_{u_1u_1} f_{u_2}\right] = \frac{f_{u_2} f_{u_1u_1}}{1+f_{u_1}^2+f_{u_2}^2}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \underbrace{\frac{1}{2}\, g^{11}\left(\cancel{g_{21,1}} + g_{11,2} - \cancel{g_{21,1}}\right)}_{\delta=1-\text{Term}} + \frac{1}{2}\, g^{12}\left(g_{22,1} + \underbrace{\cancel{g_{21,2}} - \cancel{g_{21,2}}}_{=0}\right)$$

$$= \frac{1}{g}\left[(1+f_{u_2}^2)\, f_{u_1} f_{u_1u_2} - f_{u_1} f_{u_2}^2\, f_{u_2u_1}\right]$$

$$= \frac{f_{u_1} f_{u_1u_2}}{g} = \frac{f_{u_1} f_{u_1u_2}}{1+f_{u_1}^2+f_{u_2}^2}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}\, g^{21}\, g_{11,2} + \frac{1}{2}\, g^{22}\, g_{22,1} = \frac{1}{g}\left[-f_{u_1}^2 f_{u_2}\, f_{u_1u_2} + (1+f_{u_1}^2)\, f_{u_2} f_{u_2u_1}\right]$$

$$= \frac{f_{u_2} f_{u_1u_2}}{g} = \frac{f_{u_2} f_{u_2u_1}}{1+f_{u_1}^2+f_{u_2}^2}$$

$$\Gamma^1_{22} = \frac{1}{2}\, g^{11}\left(g_{21,2} + g_{12,2} - g_{22,1} \right) + \frac{1}{2}\, g^{12} \left(g_{22,2} \right)$$

$$= \frac{f_{u_1}\, f_{u_2 u_2}}{1 + f_{u_1}^2 + f_{u_2}^2}$$

$$\Gamma^2_{22} = \frac{1}{2}\, g^{21}\left(g_{21,2} + g_{12,2} - g_{22,1} \right) + \frac{1}{2}\, g^{22} \left(g_{22,2} \right)$$

$$= \frac{f_{u_2}\, f_{u_2 u_2}}{1 + f_{u_1}^2 + f_{u_2}^2}$$

<u>Fazit:</u> $$\Gamma^l_{ik} = \frac{1}{g}\, f_{ik}\, f_l \equiv \frac{1}{g}\, f_{u_i u_k}\, f_{u_l}$$