

A2 c) Geodätische Krümmung der Loxodrome

1

$$c(s) := x(u_1(s), u_2(s))$$

$$\dot{c}(s) = x_{u_1}(u_1(s), u_2(s)) \cdot \dot{u}_1(s) + x_{u_2}(u_1(s), u_2(s)) \cdot \dot{u}_2(s) \quad (1)$$

$$|\dot{c}(s)| = \underset{g_{11}=g_{22}=0}{g_{11} \dot{u}_1^2(s) + g_{22} \dot{u}_2^2(s)} = \sin^2 u_2(s) \cdot \dot{u}_1^2(s) + \dot{u}_2^2(s) \quad (2)$$

$$\begin{aligned} \dot{u}_1(s) &= \frac{1}{\cot(x)} \left[\frac{-\sin(x)}{\sin(x)} \left(-\frac{\sqrt{2}}{4}\right) - \frac{\cos^2(x)}{\sin^2(x)} \left(-\frac{\sqrt{2}}{4}\right) \right] \quad \left(\text{mit } x := \frac{\pi - \sqrt{2}s}{4}\right) \\ &= \frac{\sqrt{2}}{4} \frac{1}{\cot(x)} \left[\frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)} \right] = \frac{1}{\sqrt{2} \cdot 2 \sin(x) \cos(x)} = \frac{1}{\sqrt{2}} \frac{1}{\sin(2x)} \\ &= \frac{1}{\sqrt{2}} \frac{1}{\sin(u_2(s))} = \dot{u}_1(s) \quad (3) \end{aligned}$$

$$\dot{u}_2(s) = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}} \quad (4)$$

$$x_{u_1} = (-\sin u_1 \sin u_2, \cos u_1 \sin u_2, 0) \quad ; \quad x_{u_2} = (\cos u_1 \cos u_2, \sin u_1 \cos u_2, -\sin u_1) \quad (5)$$

$$\begin{aligned} (2), (5) \Rightarrow \dot{c}(s) &= \begin{pmatrix} -\frac{\sin u_1(s)}{\sqrt{2}} - \frac{\cos u_1(s) \cdot \cos u_2(s)}{\sqrt{2}} \\ \frac{\cos u_1(s)}{\sqrt{2}} - \frac{\sin u_1(s) \cos u_2(s)}{\sqrt{2}} \\ \frac{\sin u_1(s)}{\sqrt{2}} \end{pmatrix} \\ (3), (4) \end{aligned}$$

$$(2), (3), (4) \Rightarrow |\dot{c}(s)| = 1 \quad \Rightarrow T(s) = \dot{c}(s) \quad \Rightarrow N(s) = \frac{\dot{T}(s)}{|\dot{T}(s)|} = \frac{\ddot{c}(s)}{|\ddot{c}(s)|} = \frac{\ddot{c}(s)}{\sqrt{2}} \quad \text{[5]}$$

$$\text{Also: } \boxed{K_g = K \langle N, n \times T \rangle = \langle \ddot{c}(s), n \times T \rangle} \quad (*)$$

$$n = -x(u_1, u_2) = (-\cos u_1 \sin u_2, -\sin u_1 \sin u_2, -\cos u_2)$$

$$\ddot{c}(s) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos(u_1(s)) \dot{u}_1(s) + \sin u_1(s) \cos u_2(s) \cdot \dot{u}_2(s) + \cos u_1(s) \sin u_2(s) \cdot \dot{u}_2(s) \\ -\sin(u_1(s)) \dot{u}_1(s) - \cos(u_1(s)) \cdot \cos(u_2(s)) \cdot \dot{u}_1(s) + \sin(u_2(s)) \cdot \sin(u_1(s)) \cdot \dot{u}_1(s) \\ \cos u_1(s) \cdot \dot{u}_2(s) \end{pmatrix}$$

$$n \times T = n \times \dot{c}(s) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin u_1(s) + \cos u_1(s) \cos u_2(s) \\ \cos u_1(s) + \cos u_2(s) \sin u_1(s) \\ -\sin u_2(s) \end{pmatrix}$$

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 $\cos^2 + \sin^2 = 1$

$$\begin{aligned} \Rightarrow \mathcal{K}_g &= \langle \ddot{c}, n \times T \rangle \\ &= -\cos u_2(s) \cdot \ddot{u}_2(s) \\ &= \frac{1}{r} \left(\underbrace{\left[\cos u_1(s) \cdot \sin u_2(s) \cdot \ddot{u}_1(s) - \cos^2 u_1(s) \cos u_2(s) \ddot{u}_2(s) - \sin^2 u_1(s) \cos u_2(s) \ddot{u}_1(s) \right.}_{\text{wavy}} \right. \\ &\quad \left. + \underbrace{\cos u_1(s) \cdot \sin u_2(s) \cos^2 u_2(s) \cdot \ddot{u}_2(s)}_{\text{dashed}} - \underbrace{\cos u_1(s) \cdot \sin u_2(s) \cdot \sin^2 u_2(s) \ddot{u}_2(s)}_{\text{wavy}} \right. \\ &\quad \left. + \underbrace{\cos^3 u_1(s) \sin u_2(s) \cos u_2(s) \cdot \ddot{u}_2(s)}_{\text{dashed}} \right) \\ &\quad + \underbrace{\left[-\cos u_1(s) \sin u_2(s) \ddot{u}_1(s) - \sin^2 u_1(s) \cos u_2(s) \ddot{u}_1(s) - \cos^2 u_1(s) \cos u_2(s) \ddot{u}_1(s) \right]}_{\text{wavy}} \\ &\quad = -\cos u_2(s) \cdot \ddot{u}_2(s) \\ &\quad - \underbrace{\cos^2 u_1(s) \cos u_2(s) \sin u_2(s) \cdot \ddot{u}_1(s)}_{\text{dashed}} + \underbrace{\cos u_1(s) \sin u_1(s) \sin u_2(s) \ddot{u}_2(s)}_{\text{wavy}} \\ &\quad + \underbrace{\cos u_1(s) \sin^2 u_1(s) \sin u_2(s) \ddot{u}_2(s)}_{\text{dashed}} \\ &\quad + \underbrace{\left[-\sin u_2(s) \cos u_2(s) \ddot{u}_1(s) \right]}_{\text{wavy}} \end{aligned}$$

Beachte:

$$\begin{aligned} \text{wavy} + \text{wavy} &= 0 \\ \text{---} + \text{---} &= 0 \\ \text{mw} + \text{mw} &= 0 \\ \text{dashed} + \text{dashed} &= \sin u_2(s) \cos u_2(s) \ddot{u}_2(s) \Rightarrow \text{dashed} + \text{dashed} + \text{wavy} = 0 \end{aligned}$$

$$\Rightarrow \mathcal{K}_g = \frac{1}{r} \left(-2 \cos u_2(s) \cdot \ddot{u}_2(s) \right) = -\cos u_2(s) \cdot \ddot{u}_2(s)$$

$$\stackrel{(3)}{=} -\frac{\cos u_2(s)}{r \sin u_2(s)} = -\frac{1}{r} \cot u_2(s) = \cos \theta(s) \cdot \mathcal{K}_g(\text{Buckkreis})$$

Also: $\frac{\mathcal{K}_g(\text{Lorochene})}{\mathcal{K}_g(\text{Buckkreis})} = \cos \theta(s) = \cot \theta(s)$ (wobei $\theta(s)$ der Schnittwinkel zwischen Lorochene und Buckkreis ist)