A2 c) Geodätische Krümmung der Loxodrome

$$c(s) := x(u_1(s), u_2(s))$$

$$\dot c(s) = x_{u_1}(u_1(s), u_2(s)) \cdot \dot u_1(s) + x_{u_2}(u_1(s), u_2(s)) \cdot \dot u_2(s) \qquad (1)$$

$$|\dot c(s)| \underset{g_{12}=g_{21}=0}{=} g_{11}\, \dot u_1^2(s) + g_{22}\, \dot u_2^2(s) = \sin^2 u_2(s) \cdot \dot u_1^2(s) + \dot u_2^2(s) \qquad (2)$$

$$\dot u_1(s) = \frac{1}{\cot(x)} \left[-\frac{\sin(x)}{\sin(x)} \left(-\frac{\sqrt{2}}{4}\right) - \frac{\cos^2(x)}{\sin^2(x)} \left(-\frac{\sqrt{2}}{4}\right) \right] \qquad \left(\text{mit } x := \frac{\pi - \sqrt{2}\, s}{4}\right)$$

$$= \frac{\sqrt{2}}{4} \frac{1}{\cot(x)} \left[\frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)} \right] = \frac{1}{\sqrt{2} \cdot 2 \sin(x) \cos(x)} = \frac{1}{\sqrt{2}} \frac{1}{\sin(2x)}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\sin(u_2(s))} = \dot u_1(s) \qquad (3)$$

$$\dot u_2(s) = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}} \qquad (4)$$

$$x_{u_1} = (-\sin u_1 \sin u_2,\ \cos u_1 \sin u_2,\ 0) \quad ; \quad x_{u_2} = (\cos u_1 \cdot \cos u_2,\ \sin u_1 \cos u_2,\ -\sin u_2) \quad (5)$$

(2), (5), (3), (4) $\Rightarrow$

$$\dot c(s) = \begin{pmatrix} -\frac{\sin u_1(s)}{\sqrt{2}} - \frac{\cos u_1(s) \cdot \cos u_2(s)}{\sqrt{2}} \\ \frac{\cos u_1(s)}{\sqrt{2}} - \frac{\sin u_1(s) \cos u_2(s)}{\sqrt{2}} \\ \frac{\sin u_2(s)}{\sqrt{2}} \end{pmatrix}$$

(2), (3), (4) $\Rightarrow |\dot c(s)| = 1 \Longrightarrow T(s) = \dot c(s) \Rightarrow N(s) = \frac{\dot T(s)}{|\dot T(s)|} = \frac{\ddot c(s)}{|\ddot c(s)|} = \frac{\ddot c(s)}{\kappa(s)}$

Also: $\boxed{\kappa_g = \kappa \langle N, n \times T \rangle = \langle \ddot c(s), n \times T \rangle}$ $(*)$

$$n = -x(u_1, u_2) = (-\cos u_1 \sin u_2,\ -\sin u_1 \sin u_2,\ -\cos u_2)$$

$$\ddot c(s) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos(u_1(s))\, \dot u_1(s) + \sin u_1(s) \cos u_2(s) \cdot \dot u_1(s) + \cos u_1(s) \sin u_2(s) \cdot \dot u_2(s) \\ -\sin(u_1(s))\, \dot u_1(s) - \cos(u_1(s)) \cdot \cos(u_2(s)) \cdot \dot u_1(s) + \sin(u_1(s)) \cdot \sin(u_2(s)) \cdot \dot u_2(s) \\ \cos u_1(s) \cdot \dot u_1(s) \end{pmatrix}$$

$$n \times T = n \times \dot c(s) \underset{\cos^2 + \sin^2 = 1}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin u_1(s) + \cos u_1(s) \cos u_2(s) \\ \cos u_1(s) + \cos u_2(s) \sin u_1(s) \\ -\sin u_2(s) \end{pmatrix}$$

$$\Rightarrow \quad \kappa_g = \langle \ddot{c}, n \times T \rangle$$

$$= \frac{1}{2}\Big(\Big[\cos u_1(s) \cdot \sin u_1(s) \cdot \dot{u}_1(s) \underbrace{- \cos^2 u_1(s) \cos u_2(s)\, \dot{u}_1(s) - \sin^2 u_1(s) \cos u_2(s)\, \dot{u}_1(s)}_{= -\cos u_2(s) \cdot \dot{u}_1(s)}$$

$$+ \cos u_1(s) \cdot \sin u_1(s)\, \cos^2 u_2(s) \cdot \dot{u}_1(s) - \cos u_1(s) \cdot \sin u_2(s) \cdot \sin u_1(s)\, \dot{u}_2(s)$$

$$+ \cos^2 u_1(s)\, \sin u_2(s)\, \cos u_2(s) \cdot \dot{u}_2(s) \Big]$$

$$+ \Big[-\cos u_1(s)\, \sin u_1(s)\, \dot{u}_1(s) \underbrace{- \sin^2 u_1(s) \cos u_2(s)\, \dot{u}_1(s) - \cos^2 u_1(s) \cos u_2(s)\, \dot{u}_1(s)}_{= -\cos u_2(s) \cdot \dot{u}_1(s)}$$

$$- \cos^2 u_2(s)\, \cos u_1(s)\, \sin u_1(s) \cdot \dot{u}_1(s) + \cos u_1(s)\, \sin u_1(s)\, \sin u_2(s)\, \dot{u}_2(s)$$

$$+ \cos u_2(s)\, \sin^2 u_1(s)\, \sin u_2(s)\, \dot{u}_2(s) \Big]$$

$$+ \Big[-\sin u_2(s)\, \cos u_2(s)\, \dot{u}_2(s) \Big] \Big)$$

Bemerkung:

$+ = 0$

$+ = 0$

$+ = 0$

$+ = \sin u_2(s) \cos u_2(s)\, \dot{u}_2(s) \Rightarrow + + = 0$

$$\Rightarrow \quad \kappa_g = \frac{1}{2}\Big(-2 \cos u_2(s) \cdot \dot{u}_1(s) \Big) = -\cos u_2(s) \cdot \dot{u}_1(s)$$

$$\overset{(3)}{=} - \frac{\cos u_2(s)}{\sqrt{2}\, \sin u_2(s)} = -\frac{1}{\sqrt{2}} \cot u_2(s) = \cos\Theta(s) \cdot \kappa_g(\text{Breitenkreis})$$

Also: $\dfrac{\kappa_g(\text{Loxodrome})}{\kappa_g(\text{Breitenkreis})} = \cos(\Theta(s)) = \text{const.}$ (wobei $\Theta(s)$ der Schnittwinkel zwischen Loxodrome und Breitenkreis ist)