Lie groups and Lie algebras
Winter semester 18/19

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3. Exercise sheet

This sheet will be discussed on 09.11.2018

Exercise 1

Show that a connected Lie group is generated by any open neighbourhood of the identity element.

Exercise 2

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\|\cdot\|$ be an arbitrary norm on $\mathfrak{g}$. Show that for $X, Y \in D := \left\{X \in \mathfrak{g} \;\middle|\; \|X\| \leq 1\right\}$

$$\exp\left(-\sqrt{t}X\right)\exp\left(-\sqrt{t}Y\right)\exp\left(\sqrt{t}X\right)\exp\left(\sqrt{t}Y\right) = \exp\left(t[X,Y] + O\left(t^{3/2}\right)\right).$$

Exercise 3

The Heisenberg-group H is the following 3-dimensional Lie subgroup of $\mathrm{SL}(3,\mathbb{R})$:

$$H := \left\{\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R}\right\}$$

a) Show that the Lie algebra $\mathfrak{h}$ of H is given by

$$\mathfrak{h} := \left\{\begin{pmatrix}0 & u & w\\ 0 & 0 & v\\ 0 & 0 & 0\end{pmatrix} \;\middle|\; u, v, w \in \mathbb{R}\right\}.$$

b) Let $A, B \in \mathfrak{h}$ and $t \in \mathbb{R}$ sufficiently small. Determine a $C(t) \in \mathfrak{h}$ such that

$$e^{tA}e^{tB} = e^{C(t)}.$$