3. Exercise sheet
This sheet will be discussed on 09.11.2018

Exercise 1
Show that a connected Lie group is generated by any open neighbourhood of the identity element.

Exercise 2
Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $\| \cdot \|$ be an arbitrary norm on $\mathfrak{g}$. Show that for $X, Y \in D := \{ X \in \mathfrak{g} \mid \|X\| \leq 1 \}$

$$\exp(-\sqrt{t}X) \exp(-\sqrt{t}Y) \exp(\sqrt{t}X) \exp(\sqrt{t}Y) = \exp(t[X, Y] + O(t^{3/2})) .$$

Exercise 3
The Heisenberg-group $H$ is the following 3-dimensional Lie subgroup of $\text{SL}(3, \mathbb{R})$:

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} .$$

a) Show that the Lie algebra $\mathfrak{h}$ of $H$ is given by

$$\mathfrak{h} := \left\{ \begin{pmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \mid u, v, w \in \mathbb{R} \right\} .$$

b) Let $A, B \in \mathfrak{h}$ and $t \in \mathbb{R}$ sufficiently small. Determine a $C(t) \in \mathfrak{h}$ such that

$$e^{tA}e^{tB} = e^{C(t)} .$$