Karlsruher Institut für Technologie

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Riemannian Geometry **Summer Term 2011**

Exercise Sheet 02 **20.04.2011**

Exercise 1

a) Let M and N be smooth manifolds. Show that $M \times N$ admits a smooth structure too.

b) Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be submanifolds. Show that $M \times N$ is a submanifold of $\mathbb{R}^{m+n}$.

Exercise 2

Let $T := \left\{ \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a + r\cos(t) \\ 0 \\ r\sin(t) \end{pmatrix} \,\middle|\, t, \varphi \in \mathbb{R} \right\}$ for $a, r > 0$.

a) Draw a picture of T and convince yourself that calling T a torus is appropriate.

b) Show that T is a submanifold of $\mathbb{R}^3$.

c) Show that $S^1 \times S^1$ amd T are diffeomorphic.

Exercise 3

The set of all one dimensional subspaces of $\mathbb{C}^2$ is the complex projective space $\mathbb{P}^1(\mathbb{C})$. One obtains $\mathbb{P}^1(\mathbb{C})$ as the set of equivalence classes given by the following equivalence relation on $\mathbb{C}^2 - \{0\}$: $(z_1, z_2) \sim (z_1', z_2')$ if there exists $\lambda \in \mathbb{C}^*$, such that $(z_1, z_2) = \lambda(z_1', z_2')$. Hence $\mathbb{P}^1(\mathbb{C})$ is a topological space with the quotient topology inherited from $\mathbb{C}^2 - \{0\}$. Let $[z_1 : z_2]$ denote the equivalence class of (z_1, z_2) in $\mathbb{P}^1(\mathbb{C})$.

a) Show that the maps $\phi_1 : U_1 \to \mathbb{C}$, $[z : 1] \mapsto z$ with $U_1 := \{[z : 1] \mid z \in \mathbb{C}\}$ and $\phi_2 : U_2 \to \mathbb{C}$, $[1 : z] \mapsto z$ with $U_2 := \{[1 : z] \mid z \in \mathbb{C}\}$ are homeomorphisms from the open sets U_i of $\mathbb{P}^1(\mathbb{C})$ to $\mathbb{C}$.

b) Compute the sets $V_1 = \phi_1(U_1 \cap U_2)$ and $V_2 = \phi_2(U_1 \cap U_2)$ and the coordinate change $\phi_2 \circ \phi_1^{-1} : V_1 \to V_2$.

c) Construct a diffeomorphism from $\mathbb{P}^1(\mathbb{C})$ to S^2.
Hint: Look at the stereographic projections of S^2 (with respect to the south and the north pole) and their inverse maps $\psi_1 : \mathbb{C} \to S^2 - \{N\}$ and $\psi_2 : \mathbb{C} \to S^2 - \{S\}$ respectively.
