Karlsruher Institut für Technologie

Institute for Algebra and Geometry
HDoz.Dr. Oliver Baues
Dipl.Math. Johannes Riesterer

Riemannian Geometry **Summer Term 2011**

Exercise Sheet 04 **04.05.2011**

Exercise 1

Let M be a smooth manifold. Show that the tangent bundle TM (with its canonical smooth structure) is an orientable manifold.

Exercise 2

Let $X, Y : U \to \mathbb{R}^n$ be smooth vector fields on an open subset $U \subset \mathbb{R}^n$. Let $[X, Y]$ be the smooth vector field which satisfies $L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X$. $[X, Y]$ is called the Lie bracket. Show that

a) $[X, Y] = d_X Y - d_Y X$ (equivalently $[X, Y] = \sum_{i=1}^{n} Z^i \frac{\partial}{\partial x_i}$ with $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$, $Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial x^i}$ and $Z^i = \sum_{i=1}^{n} \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right)$).

b) $[X, Y] = -[Y, X]$,

c) $[X, fY] = (L_X f)Y + f[X, Y]$, where $f \in \mathcal{C}^\infty(U)$.

Exercise 3

Let M be a smooth manifold. Show that the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

holds for all X, Y, Z smooth vector fields on M.

Exercise 4

For $x_0 \in \mathbb{R}^n$, $0 < a < b \in \mathbb{R}$ construct a $\mathcal{C}^\infty$–function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases} f(x) \geq 1 & \text{for } x \in \overline{B_a(x_0)} \, , \\ f(x) = 0 & \text{for } x \notin B_b(x_0) \, . \end{cases}$$

f is called a test function.

Hint: Use the $\mathcal{C}^\infty$–function $f_r : \mathbb{R} \to \mathbb{R}$, $f_r(x) = \begin{cases} e^{\frac{1}{x^2 - r^2}} & \text{for } |x| < r, \\ 0 & \text{for } |x| \geq r \, . \end{cases}$