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Riemannian Geometry **Summer Term 2011**

Exercise Sheet 06 **18.05.2011**

Let $\pi : E \to M$ and $\pi' : E' \to M$ be vector bundles over a manifold M. Define a bundle morphism $f : E \to E'$ to be a smooth map such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & E' \\ & {}_{\pi}\searrow \quad \swarrow_{\pi'} & \\ & M & \end{array}$$

commutes and $f_p := f_{|E_p} : E_p \to E'_p$ is linear for all $p \in M$. If E is contained in E' and E_p is a sub-vector-space of E'_p for all $p \in M$ then E is called a sub-bundle of E'.

Exercise 1

a) Define $\ker(f) := \bigcup_{p\in M} \ker(f_p)$ and similar $\operatorname{im}(f) := \bigcup_{p\in M} \operatorname{im}(f_p)$. If f_p has constant rank (as a function on p) then $\ker(f)$ is a sub-bundle of E and $\operatorname{im}(f)$ is a sub-bundle of E'.

b) Show that there exists a vector bundle $E \oplus E'$ over M with fiber $(E \oplus E')_p = E_p \oplus E'_p$ for all $p \in M$. This bundle is called the Whitney sum of E and E'.

Exercise 2

Let

$$0 \longrightarrow E' \xrightarrow{\ i\ } E \xrightarrow{\ p\ } E'' \longrightarrow 0$$

be a short exact sequence of vector bundles over a manifold M. That is i and p are morphisms of vector bundles such that $\ker(p) = \operatorname{im}(i)$, p is surjective and i is injective.

a) Show that there exists a bundle morphism $\sigma : E'' \to E$ with $p \circ \sigma = \operatorname{id}$.
Hint: Use a bundle metric on E.

b) Show that $E \cong E' \oplus E''$.

Exercise 3

Show that TS^2 does not admit a smooth family $x \mapsto \langle\cdot,\cdot\rangle_x$ of symmetric bilinear forms

$$\langle\cdot,\cdot\rangle_x : T_xS^2 \times T_xS^2 \to \mathbb{R}$$

of signature $(1, 1)$.

Hint: Construct a nowhere vanishing vector field on S^2 under the assumption that such a family exists.
