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Riemannian Geometry **Summer Term 2011**

Exercise Sheet 07/08 **25.05.2011**

Exercise 1

Let V be a real vector space. An alternating $k-$form ω on V is defined as a multilinear map

$$\omega : \underbrace{V \times \cdots \times V}_{k-\text{times}} \to \mathbb{R}$$

with the property that $\omega(v_{\tau(1)}, \ldots, v_{\tau(k)}) = \operatorname{sgn}(\tau)\omega(v_1, \ldots, v_k)$ for all permutations τ. The set of all alternating $k-$forms is denoted by $(\bigwedge^k V)^*$.

i) Show that $(\bigwedge^k V)^*$ is a vector space.

ii) Let $E \to M$ be a vector bundle over a manifold. Show that there exists a vector bundle $\pi : (\bigwedge^k E)^* \to M$ with fiber $\pi^{-1}(p) = (\bigwedge^k E_p)^*$ for all $p \in M$. For $k = 1$, $(\bigwedge^1 E)^* = E^*$ is called the dual bundle.

Exercise 2

Let M be an $n-$dimensional smooth manifold. A smooth, nowhere vanishing section $\omega : M \to (\bigwedge^n TM)^*$ is called a volume form on M. For a smooth map $f : M \to N$ and a section $\sigma : N \to (\bigwedge^n TN)^*$ define the pullback $f^*\sigma$ by $f^*\sigma_p(v_1, \ldots, v_n) := \sigma_p(df_p(v_1), \ldots, df_p(v_n))$.

a) Let ω and ω' be two volume forms on M. Show that there exists a smooth function $g \in \mathcal{C}^\infty(M)$ with $\omega'_p = g(p)\omega_p$.

b) Show that S^n admits a volume form.
Hint: Use the standard volume form $v_1 \wedge \ldots \wedge v_n$ on the vectorspace $\mathbb{R}^{n+1}$ to induce a volume form on S^n.

c) Let $f : M \to N$ be a local diffeomorphism and ω a volume form on N. Then M admits a volume form too.

Exercise 3

Show that the natural map $\pi : S^n \to \mathbb{P}^n(\mathbb{R}), (x_0, \ldots, x_n) \mapsto [x_0 : \ldots : x_n]$ is a local diffeomorphism. Every fiber is of the form $\pi^{-1}([x]) = \pm\frac{x}{||x||} \subset S^n$, that is $\mathbb{P}^n(\mathbb{R}) \cong S^n/\{\pm 1\}$.

Exercise 4

Show that $\mathbb{P}^n(\mathbb{R})$ admits a volume form if and only if n is odd.
