Karlsruher Institut für Technologie

Institute for Algebra and Geometry
HDoz.Dr. Oliver Baues
Dipl.Math. Johannes Riesterer

Riemannian Geometry

Summer Term 2011

Exercise Sheet 09

08.06.2011

Exercise 1

a) Let (M, g), (N, h) be two riemannian manifolds and $\phi : M \to N$ be a diffeomorphism. Show that ϕ is an isometry if and only if the length of any smooth curve $\gamma : I \to M$ is invariant under ϕ, that is $L_g(\gamma) = L_h(\phi \circ \gamma)$.

b) Show that the isometry group of a riemannian manifold as defined in the lecture is indeed a group with respect to the composition of maps.

Exercise 2

Let $(\mathbb{H}^2, g)$ be the upper half-space defined by $\mathbb{H}^2 := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and

$$(g_{ij}(z))_{i,j} := \frac{1}{\mathrm{Im}(z)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with respect to the chart id of $\mathbb{H}^2$. Show that for all $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ the map

$$z \mapsto \frac{az + b}{cz + d}$$

is an isometry of $\mathbb{H}^2$.

Exercise 3

For $a \neq 0$, $\Phi : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$, $(u, v) \mapsto (a \cosh v \cos u, a \cosh v \sin u, av)$ is called *Katenoid* and $\Psi : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$, $(u, v) \mapsto (v \cos u, v \sin u, au)$ is called *Helikoid*.

Convince yourself that the Katenoid and the Helikoid are submanifolds of $\mathbb{R}^3$ and show that they are locally isometric (with respect to the induced metrics from $\mathbb{R}^3$).

Exercise 4

Let M be a smooth manifold and X be a vector field on M with compact support. Show that X is complete.
