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Riemannian Geometry

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Exercise Sheet 11

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Exercise 1

A Lie group is a group whose underlying set G is a smooth manifold such that the maps

$$\mu : G \times G \to G,\ (g,h) \mapsto gh \ \text{ and } \ \iota : G \to G,\ g \mapsto g^{-1}$$

are smooth. Moreover let $L_g(h) = gh$ denote left multiplication and $R_g(h) = hg$ right multiplication by a fixed element $g \in G$. A vector field $X \in Vect(G)$ is called left invariant if

$$L_g^* X = X \text{ for all } g \in G\ .$$

Denote by $\mathfrak{g}$ the vector space of left invariant vector fields which is called the Lie algebra of G. Show that:

(a) $\mathfrak{g}$ is a vector space of dimension $\dim(\mathfrak{g}) = \dim(G)$.
Hint: Show that $\mathfrak{g}$ is isomorphic to T_eG where e is the identity element of G.

(b) $\mathfrak{g}$ is closed under the Lie bracket, that is $[X,Y] \in \mathfrak{g}$ for all $X, Y \in \mathfrak{g}$.

Exercise 2

Let I denote the identity matrix. Show that:

(a) $\mathrm{O}(n) := \{A \in \mathbb{R}^{n\times n} \mid AA^t = I\}$ is a compact Lie group.

(b) $T_I\mathrm{O}(n) := \{X \in \mathbb{R}^{n\times n} \mid X = -X^t\}$. What is $T_A\mathrm{O}(n)$, for $A \in \mathrm{O}(n)$?

(c) $\mathrm{SO}(n) := \mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R})$ is the connected component of $\mathrm{O}(n)$ which includes the identity matrix.

(d) $\mathrm{SO}(n)$ is a Lie group, $T_A\mathrm{SO}(n) = T_A\mathrm{O}(n)$ for all $A \in \mathrm{SO}(n)$.

Exercise 3

Define $\langle X, Y\rangle_A := \mathrm{trace}(A^{-1}X(A^{-1}Y)^t),\ A \in \mathrm{SO}(n);\ X, Y \in T_A\mathrm{SO}(n)$. Show that:

(a) $\langle\cdot,\cdot\rangle$ defines a Riemannian metric on $\mathrm{SO}(n)$.

(b) $\langle\cdot,\cdot\rangle$ is biinvariant, that is $L_A^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle$ and $R_A^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle$ for all $A \in \mathrm{SO}(n)$.

(c) Let X, Y be left invariant vector fields on $\mathrm{SO}(n)$. Then

$$D_X Y = \frac{1}{2}[X,Y]\ ,$$

where D is the Levi-Civita connection of $\langle\cdot,\cdot\rangle$.

(d) The geodesics on $(\mathrm{SO}(n), \langle\cdot,\cdot\rangle)$ through I are precisely the curves

$$c_X : \mathbb{R} \to \mathrm{SO}(n),\ c_X(t) = e^{tX} := \sum_{j=0}^{\infty} \frac{(tX)^j}{j!}\ .$$