Institute for Algebra and Geometry
HDoz.Dr. Oliver Baues
Dipl.Math. Johannes Riesterer

Riemannian Geometry

Summer Term 2011

Solution to Exercise Sheet 11

11.07.2011

Exercise 1

A Lie group is a group whose underlying set G is a smooth manifold such that the maps

$$\mu : G \times G \to G,\ (g,h) \mapsto gh \ \text{ and } \ \iota : G \to G,\ g \mapsto g^{-1}$$

are smooth. Moreover let $L_g(h) = gh$ denote left multiplication and $R_g(h) = hg$ right multiplication by a fixed element $g \in G$. A vector field $X \in Vect(G)$ is called left invariant if

$$L_g^* X = X \text{ for all } g \in G\ .$$

Denote by $\mathfrak{g}$ the vector space of left invariant vector fields which is called the Lie algebra of G. Show that:

(a) $\mathfrak{g}$ is a vector space of dimension $\dim(\mathfrak{g}) = \dim(G)$.
Hint: Show that $\mathfrak{g}$ is isomorphic to T_eG where e is the identity element of G.

(b) $\mathfrak{g}$ is closed under the Lie bracket, that is $[X, Y] \in \mathfrak{g}$ for all $X, Y \in \mathfrak{g}$.

Solution:

a) First recall the definition $(L_g)_*X = dL_gX$ and $(L_g)^*X = (dL_g)^{-1}X)$ which implies that $(L_g)_*$ is a linear map. Since $(L_g)^* = (L_{g^{-1}})_*$, a vector field is left invariant, if and only if $(L_g)_*X = X$ for all $g \in G$. By linearity of $(L_g)_*$ for $X, Y \in \mathfrak{g}$ we have

$$(L_g)_*(\lambda X + \mu Y) = (L_g)_*\lambda X + (L_g)_*\mu Y = \lambda(L_g)_*X + \mu(L_g)_*Y = \lambda X + \mu Y$$

and $(L_g)_*0 = 0$ (where 0 denotes the vector field $X(g) = 0$ for all $g \in G$) which implies that $\mathfrak{g}$ is a vector space. Moreover for a left invariant vector field $X \in \mathfrak{g}$

$$X(gh) = X(L_gh) = (L_g)_*(X(h)) = (dL_g)|_h(X(h))$$

by definition and hence for $h = e$ we get $X(g) = (dL_g)|_e(X(e))$, where $e \in G$ is the identity element of G. This means that a left invariant vector field is determined by its value at the identity. This may lead to the following definition of a map

$$\begin{aligned} \tau : T_eG &\to \mathfrak{g} \\ v &\mapsto X_v\ , \end{aligned}$$

where X_v is the vector field given by

$$X_v(g) = dL_g|_e v\ .$$

We now have to check several things, namely that X_v is left invariant and that the map τ is bijective (linearity is evident). By definition of X_v we compute

$$X_v(gh) = dL_{gh}|_e v = dL_{L_g(h)}|_e v = (dL_g)_{|h} dL_h|_e v = dL_g|_h(X_v(h))$$

which implies left invariance. The map τ is injective, because for $v \neq w \in T_eG$ we have $X_v(g) := dL_g|_e v \neq dL_g|_e w =: X_w(g)$ since dL_g is an isomorphism of vector spaces for all $g \in G$. The map τ is surjective, because for $X \in \mathfrak{g}$ and $v = X(e)$ we have

$$X(g) = dL_g|_e v = X_v(g)\ .$$

b) One can show this exercise more elementary, but i want to use a more general, useful lemma, which holds for all smooth manifolds.

Lemma. *Let $\phi : M \to N$ be a diffeomorphism of smooth manifolds, then $\phi_*[X,Y] = [\phi_* X, \phi_* Y]$.*

Proof. We prove it for the corresponding derivations. Therefore note that $(\phi_* X)(f) = X(f \circ \phi) \circ \phi^{-1}$, which follows directly from definition. We compute

$$\begin{aligned}[\phi_* X, \phi_* Y](f) &= (\phi_* X)(\phi_* Y(f)) - (\phi_* Y)(\phi_* X(f)) = (\phi_* X)(Y(f \circ \phi) \circ \phi^{-1}) - (\phi_* Y)(X(f \circ \phi) \circ \phi^{-1}) \\ &= X(Y(f \circ \phi) \circ \phi^{-1} \circ \phi) \circ \phi^{-1} - Y(X(f \circ \phi) \circ \phi^{-1} \circ \phi) \circ \phi^{-1} \\ &= X(Y(f \circ \phi)) \circ \phi^{-1} - Y(X(f \circ \phi)) \circ \phi^{-1} = \phi_*(XY - YX)(f) \\ &= \phi_*[X,Y]\end{aligned}$$

□

Since L_g is a diffeomorphism by definition of a Lie group (L_g is bijective and smooth and $(L_g)^{-1} = L_{g^{-1}}$) the lemma implies

$$(L_g)_*[X,Y] = [(L_g)_* X, (L_g)_* Y] = [X,Y]$$

for all left invariant vector-fields $X, Y \in \mathfrak{g}$ and hence $[X,Y]$ is a left invariant vector field.

Exercise 2

Let I denote the identity matrix. Show that:

(a) $\mathrm{O}(n) := \{A \in \mathbb{R}^{n \times n} \mid AA^t = I\}$ is a compact Lie group.

(b) $T_I\mathrm{O}(n) := \{X \in \mathbb{R}^{n \times n} \mid X = -X^t\}$. What is $T_A\mathrm{O}(n)$, for $A \in \mathrm{O}(n)$?

(c) $\mathrm{SO}(n) := \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$ is the connected component of $\mathrm{O}(n)$ which includes the identity matrix.

(d) $\mathrm{SO}(n)$ is a Lie group, $T_A\mathrm{SO}(n) = T_A\mathrm{O}(n)$ for all $A \in \mathrm{SO}(n)$.

Solution:

a) As topology on $\mathrm{O}(n)$ we take the subspace topology induced from the standard topology on $\mathbb{R}^{n \times n}$. The standard topology on $\mathbb{R}^{n \times n}$ is the standard topology on $\mathbb{R}^{n^2}$ by identifying them in the most canonical way (Take colums of the matrix and write them in one vector....). Moreover let $\mathrm{Sym}(n) := \{A \in \mathbb{R}^{n \times n} \mid A = A^t\}$ be the set of symmetric $n \times n$ matrices and define

$$\begin{aligned} f : \mathbb{R}^{n \times n} &\to \mathrm{Sym}(n) \\ A &\mapsto AA^t\ , \end{aligned}$$

which is a differentiable map and $\mathrm{O}(n) = f^{-1}(I)$. If we show that I is a regular value, then $\mathrm{O}(n)$ is a submanifold of $\mathbb{R}^{n^2}$ of dimension $\dim(\mathrm{O}(n)) = \dim(\mathbb{R}^{n^2}) - \dim(\mathrm{Sym}(n)) = n^2 - \frac{n(n+1)}{2}$, since $\dim(\mathrm{Sym}(n)) = \sum_{i=n}^{1} i = \frac{n(n+1)}{2}$.

By identifying $\mathbb{R}^{n^2}$ with $\mathbb{R}^{n \times n}$ again we can view f as a differentiable map from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{\frac{n(n+1)}{2}}$ and there we can explicitly compute the differential as known from analysis. In this way we see that $dfA(H) = HA^t + AH^t$ for all $A, H \in \mathbb{R}^{n \times n}$. For $S \in \mathrm{Sym}(n)$ and $A \in \mathrm{O}(n)$ define $H = \frac{1}{2}SA$ which yields

$$df_A(H) = \frac{1}{2}A(SA)^t + \frac{1}{2}SAA^t = S$$

since A is orthogonal. This implies that df_A is surjective and hence I is a regular value as desired. The proof, that multiplication and the inverse map are smooth is the same as for the group $\mathrm{SL}_n(\mathbb{R})$ (compare Exercise 1 on Exercise Sheet 03). Since $\mathrm{O}(n)$ is a subset of $\mathbb{R}^{n^2}$, to prove compactness it is enough to show that $\mathrm{O}(n)$ is closed and bounded by the theorem of Heine-Borel. A point in $\mathbb{R}^{n^2}$ is a closed subset and since f is continuous, $\mathrm{O}(n)$ is the preimage of a closed set and hence closed. To see that $\mathrm{O}(n)$ is bounded, we introduce a norm on $\mathbb{R}^{n \times n}$. Define $||M|| := \sqrt{\mathrm{trace}(MM^t)} = \sqrt{\sum_{i,j=1}^{n} g_{ij}^2}$ for $M \in \mathbb{R}^{n \times n}$.

This is obviously a norm. Hence $||A|| = \sqrt{n}$ for all $A \in \mathrm{O}(n)$ and $\mathrm{O}(n)$ is bounded by the ball $B(0, \sqrt{n})$, which completes the proof of compactness.
b) Let $c : (-\varepsilon, \varepsilon) \to \mathrm{O}(n)$ be a smooth curve with $c(0) = I$. Then $f \circ c(t) = I$ implies

$$0 = df_I c'(0) = c'(0)^t + c'(0)$$

and hence $T_I\mathrm{O}(n) \subset \{A \in \mathbb{R}^{n\times n} \mid A + A^t = 0\}$. Since for every symmetric matrix A the identity

$$A = \frac{1}{2}A + \frac{1}{2}A^t$$

holds, $\mathrm{Sym}(n)$ is the image of the map

$$\begin{aligned} \phi : \mathbb{R}^{n\times n} &\to \mathbb{R}^{n\times n} \\ M &\mapsto M + M^t \,. \end{aligned}$$

This implies

$$\dim(\{A \in \mathbb{R}^{n\times n} \mid A + A^t = 0\}) = \dim(\ker\phi) = n^2 - \dim\mathrm{Sym}(n) = \dim\mathrm{O}(n)$$

and so $T_I\mathrm{O}(n) = \{A \in \mathbb{R}^{n\times n} \mid A + A^t = 0\}$. The tangent space $T_A\mathrm{O}(n) = (L_A)_* T_I\mathrm{O}(n) = dL_A \cdot T_I\mathrm{O}(n) = A \cdot T_I\mathrm{O}(n)$.
c) Every matrix $A \in \mathrm{SO}(n)$ is conjugated via an orthogonal matrix O to a matrix of the form

$$OAO^t = D := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & D(\varphi_i) & \\ & & & \ddots \end{pmatrix}$$

where $D(\varphi_i) := \begin{pmatrix} cos(\varphi_i) & sin(\varphi_i) \\ -sin(\varphi_i) & cos(\varphi_i) \end{pmatrix}$ and there is only a 1 in the left upper corner, if n is odd. Now define $D(t) := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & D(t \cdot \varphi_i) & \\ & & & \ddots \end{pmatrix}$. This defines a smooth curve between $D(0) = I$ and $D(1) = D$. Since $\mathrm{SO}(n)$ is a normal subgroup of $\mathrm{O}(n)$, $O^t D(t) O : [0, 1] \to \mathrm{SO}(n)$ defines a smooth curve from I to A. Since $\mathrm{O}(n)$ is a manifold, connected implies path connected, and hence $\mathrm{SO}(n)$ is the connected component containing the identity I, which also implies **d)** (The other connected component of $\mathrm{O}(n)$ are the orthogonal matrices with determinant -1, since det is continuous and $0 \notin \mathrm{O}(n)$).

Exercise 3

Define $\langle X, Y \rangle_A := \mathrm{trace}(A^{-1}X(A^{-1}Y)^t)$, $A \in \mathrm{SO}(n)$; $X, Y \in T_A\mathrm{SO}(n)$. Show that:

(a) $\langle \cdot, \cdot \rangle$ defines a Riemannian metric on $\mathrm{SO}(n)$.

(b) $\langle \cdot, \cdot \rangle$ is bi-invariant, that is $L_A^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle$ and $R_A^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle$ for all $A \in \mathrm{SO}(n)$.

(c) Let X, Y be left invariant vector fields on $\mathrm{SO}(n)$. Then

$$D_X Y = \frac{1}{2}[X, Y] \,,$$

where D is the Levi-Civita connection of $\langle \cdot, \cdot \rangle$.

(d) The geodesics on $(\mathrm{SO}(n), \langle \cdot, \cdot \rangle)$ through I are precisely the curves

$$c_X : \mathbb{R} \to \mathrm{SO}(n),\ c_X(t) = e^{tX} := \sum_{j=0}^{\infty} \frac{(tX)^j}{j!} \,.$$

Solution:

a) Recall the definition $\text{trace}(M) = \sum_{i=1}^{n} m_{ii}$, where $M = (m_{ij})_{i,j} \in \mathbb{R}^{n\times n}$. Directly from this definition we see that $\text{trace}(M) = \text{trace}(M^t)$. Moreover recall that $(MN)^t = N^t M^t$ and that $(M^t)^t = M$ for all $M, N \in \mathbb{R}^{n\times n}$. Hence we get

$$\begin{aligned}\langle X, Y\rangle_A &:= \text{trace}(A^{-1}X(A^{-1}Y)^t) = \text{trace}((A^{-1}Y(A^{-1}X)^t)^t) = \text{trace}((A^{-1}Y(A^{-1}X)^t)\\ &= \langle Y, X\rangle_A \ ,\end{aligned}$$

which shows that $\langle X, Y\rangle_A$ is symmetric.
By definition $AA^t = I$ for all $A \in \text{SO}(n)$, which yields $A^{-1} = A^t$. With that we get $(A^{-1}X)^t = (A^t X)^t = X^t (A^t)^t = X^t A$ for all $A \in \text{SO}(n)$ and $X \in T_A\text{SO}(n)$. It is easy to see (or should be known from linear algebra) that $\text{trace}(A^{-1}XA) = \text{trace}(X)$. We compute

$$\langle X, X\rangle_A := \text{trace}(A^{-1}X(A^{-1}X)^t) = \text{trace}((A^{-1}XX^tA) = \text{trace}(XX^t) \ .$$

As already mentioned $\text{trace}(XX^t) = 0$, if and only if $X = 0$ and $\text{trace}(XX^t) > 0$ for $X \neq 0$, which implies that $\langle X, X\rangle_A$ is positive definite.
b) We compute

$$\begin{aligned}(L_A^*\langle\cdot,\cdot\rangle)(X,Y)_B &:= \langle (L_A)_*X, (L_A)_*Y\rangle_{L_A(B)} = \langle AX, AY\rangle_{AB} := \text{trace}((AB)^{-1}AX((AB)^{-1}AY)^t)\\ &= \text{trace}(B^{-1}A^{-1}AX((B^{-1}A^{-1}AY)^t) = \text{trace}(B^{-1}X((B^{-1}Y)^t)\\ &= \langle X, Y\rangle_B\end{aligned}$$

and hence $\langle X, Y\rangle_A$ is a left invariant metric. Similar calculation holds for R_g which implies that bi-invariance.
c) Let X, Y be left invariant vector fields on $\text{SO}(n)$. Since $\langle\cdot,\cdot\rangle$ is left invariant $\langle X(a), Y(a)\rangle_a = \langle (L_a)_*X(e), (L_a)_*Y(e)\rangle_a = \langle X(e), Y(e)\rangle_e$ and hence the function $a \mapsto \langle X(a), Y(a)\rangle_a$ is constant which implies $X\langle Y, Z\rangle = 0$ for all left-invariant vector fields. With that and the Koszul formula we get

$$\begin{aligned}2\langle D_X Y, Z\rangle &= \underbrace{X\langle Y, Z\rangle}_{=0} + \underbrace{Y\langle Z, X\rangle}_{=0} - \underbrace{Z\langle X, Y\rangle}_{=0}\\ &\quad - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle\\ &= -\langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle\\ &= \langle [Y, X], Z\rangle + \langle [X, Y], Z\rangle - \langle Z, [Y, X]\rangle\\ &= \langle [X, Y], Z\rangle \ .\end{aligned}$$

Since Z was an arbitrary left invariant vector field, the claim follows.
