Exercise 1

A Riemannian manifold $(M,g)$ is called a homogeneous space, if for all $x, y \in M$ there exists $\psi \in \text{Isom}(M,g)$, such that $\psi(x) = y$. Show that a homogeneous space is complete.

Exercise 2

Let $(M,g)$ be a connected Riemannian manifold and $\phi, \psi \in \text{Isom}(M,g)$ be isometries. Show that, if $d\phi_p = d\psi_p$ and $\phi(p) = \psi(p)$ for one $p \in M$, then $\phi = \psi$.

Hint: Show that the set $\{q \in M \mid \phi(q) = \psi(q)\}$ is an open and closed subset of $M$.

Exercise 3

Let $G$ be a Lie group, and let $h$ a bi-invariant metric (Compare Exercise Sheet 11 Exercise 3). Moreover let $D$ be the Levi-Civita connection for $h$. Let $g$ be the vector space of left-invariant vector fields on $G$.

a) Show that $D_X Y = \frac{1}{2}[X,Y]$ and $R(X,Y)Z = \frac{1}{4}[[X,Y]Z]$ holds for all left-invariant vector fields $X, Y \in g$.

c) Let $X_e, Y_e \in T_e G$ be orthogonal with respect to $h$. Show that for the sectional curvature at the point $e \in G$

$$K_e(X_e, Y_e) = \frac{1}{4}||\{X,Y\}(e)||_h$$

holds, where $X$ and $Y$ are the left invariant vector fields $X(e) = X_e$ and $Y(e) = Y_e$. Conclude that the sectional curvature is non-negative everywhere.

d) A Riemannian manifold is called flat, if its sectional curvatures are zero everywhere. Characterise all Lie groups with flat bi-invariant metric.

Exercise 4

Let $S^2$ be the two sphere. Show that, for every vector field $X : S^2 \to TS^2$, there exists $p \in M$ with $X(p) = 0$.


We wish all participants of the lecture happy holidays and all the best for their further studies.