Deformations and rigidity of lattices in solvable Lie groups
joint work with Benjamin Klopsch

Oliver Baues

Institut für Algebra und Geometrie, Karlsruher Institut für Technologie (KIT), 76128 Karlsruhe, Germany

June 21, 2011
Outline

- Rigidity of lattices in Lie groups
  - Rigidity theorem of Mal’cev and Saitô.
- Rigid embedding into algebraic groups
- Quantitative description of the rigidity problem
- The “zoo” of solvable Lie groups
- Unipotently connected groups
- Finiteness theorem for $D(\Gamma, G)$
- Strong rigidity and structure set
- Representation of the structure set
Let $G$ be a (connected) Lie group, $\Gamma \leq G$ a discrete subgroup.  

**Definition**  
$\Gamma$ is called a *lattice* in $G$ if $G/\Gamma$ is compact (or has finite volume).

**Example**  
$\mathbb{Z}^n \leq \mathbb{R}^n$, $\text{SL}(n, \mathbb{Z}) \leq \text{SL}(n, \mathbb{R})$.

$\Gamma$ is a discrete approximation of $G$.

**Q:** How closely related are $\Gamma$ and $G$?
Mostow strong rigidity

**Theorem**

Let $G$ and $G'$ be semisimple Lie groups of non-compact type with trivial center, not locally isomorphic to $\text{SL}(2, \mathbb{R})$, and, $\Gamma \leq G$, $\Gamma' \leq G'$ irreducible lattices. Then every isomorphism

$$\varphi : \Gamma \rightarrow \Gamma'$$

extends uniquely to an isomorphism of ambient Lie groups

$$\hat{\varphi} : G \rightarrow G'.$$
$\Gamma \leq G$ a lattice.

**Definition**

$\Gamma$ is *rigid* if for any isomorphism $\varphi : \Gamma \to \Gamma'$, where $\Gamma' \leq G$ is a lattice, there exists an extension $\hat{\varphi} : G \to G$.

**Definition**

$\Gamma$ is *weakly rigid* if for any automorphism $\varphi : \Gamma \to \Gamma$ there exists an extension $\hat{\varphi} : G \to G$.

Important examples in the context of solvable Lie groups:

Auslander 1960, Milovanov 1973, Starkov 1994
Rigidity theorem of Mal’cev (1949) and Saitô (1957)

Theorem

Let $G$ and $G'$ be simply connected nilpotent solvable Lie groups of real type and $\Gamma \leq G$, $\Gamma' \leq G'$ lattices. Then every isomorphism $\varphi : \Gamma \to \Gamma'$ extends uniquely to an isomorphism of ambient Lie groups $\hat{\varphi} : G \to G'$.

In particular, $\Gamma$ a lattice in a simply connected solvable Lie group of real type. Then $\Gamma$ is rigid in $G$. 
Let $\Gamma$ be polycyclic, torsionfree.

**Theorem (Existence)**

There exist a $\mathbb{Q}$-defined linear algebraic group $A$ and an embedding $\iota : \Gamma \hookrightarrow A$ such that $\iota(\Gamma) \subseteq A_{\mathbb{Q}}$ and

(i) $\iota(\Gamma)$ is Zariski-dense in $A$,

(ii) $A$ has a strong unipotent radical, i.e. $C_A(\text{Rad}_u(A)) \subseteq \text{Rad}_u(A)$,

(iii) $\dim \text{Rad}_u(A) = \text{rk } \Gamma$.

The group $A = A(\Gamma)$ is called an *algebraic hull* for $\Gamma$. 
Proposition (Rigidity of the algebraic hull)

Let $\Gamma \leq A = A(\Gamma)$, and $\Gamma' \leq B = A(\Gamma')$ be $\mathbb{Q}$-defined algebraic hulls. Then every isomorphism

$$\varphi : \Gamma \rightarrow \Gamma'$$

extends uniquely to a $\mathbb{Q}$-defined isomorphism of algebraic groups

$$\Phi : A \rightarrow B.$$
The space of lattice embeddings

$$\mathcal{X}(\Gamma, G) := \{\varphi: \Gamma \hookrightarrow G \mid \varphi(\Gamma) \text{ is a lattice in } G\}$$

The deformation space of $\Gamma$ is

$$\mathcal{D}(\Gamma, G) = \text{Aut}(G) \backslash \mathcal{X}(\Gamma, G).$$

**Definition**

$\Gamma$ is called deformation rigid if

$$\mathcal{D}(\Gamma, G)_0 = \text{Aut}(G)_0 \backslash \mathcal{X}(\Gamma, G)_0 = \{\ast\}.$$ 

**Examples:**

- $\mathbb{Z}^3$ is not deformation rigid in $\widetilde{E}(2)$.
- Milovanov 1973: non-deformation rigid $\Gamma$ in $G$ of type (E), $\dim G = 5$. 
Auslander 1973/Starkov 1994: Classification via the eigenvalues $\lambda$ of the adjoint representation $\text{Ad}: G \to \text{GL}(g)$.

- **nilpotent** $N$
  $(\mathbb{R}^n, +)$, Heisenberg-group $H_3(\mathbb{R})$.
- **real type** $R$
  3-dimensional unimodular group $\text{Sol}$.
- **exponential type** $E$
  no $\lambda$ on the unit circle, except 1
- **type A**
  all $\text{Ad}(g)$ are either unipotent, or have a $\lambda$ with $|\lambda| \neq 1$
  First example by Auslander 1960, dim $G = 5$.
- **type I** (*imaginary type*)
  all $\lambda$ on the unit circle
- **mixed**

$$N \subset R \subset E \subset A, \ I \cap A = N$$
Definition (F. Grunewald, D. Segal\textsuperscript{1})

Let $G \leq \text{GL}(N, \mathbb{R})$ be a solvable Lie subgroup. Then $G$ is called \textit{unipotently connected} if $G \cap u(G)$ is connected.

We say $G$ is unipotently connected if it is unipotently connected as a subgroup of its algebraic hull $A_G$.

Proposition

\textit{Every $\Gamma$ has a finite index subgroup $\Gamma'$ which is a Zariski-dense lattice in a unipotently connected group $G'$}.

The following are equivalent:

1. $G$ is unipotently connected.
2. $G$ is of type $A$ in the A-S classification.

\textsuperscript{1}On affine crystallographic groups, JDG 40, 1994
Theorem (A)

Let $G$ be simply connected, and unipotently connected. Then, for every Zariski-dense lattice $\Gamma$ of $G$, the deformation space $D(\Gamma, G)$ is finite.

Both assumptions (u-connected) and Zariski-dense are necessary.

Example

There exists a pair $(G, \Gamma)$, $G$ is of mixed type, $\Gamma \leq G$ is Zariski-dense, $\dim G = 12$, $\dim N(G) = 8$, $\rk \Fitt(\Gamma) = 10$, such that $D(\Gamma, G)$ is countably infinite.
Let $\Gamma$ be a lattice in a simply connected, solvable Lie group $G$.

**Corollary (1)**

*There exists a finite index subgroup $\Gamma'$ of $\Gamma$ which embeds as a Zariski-dense lattice into $G'$ such that the deformation space $D(\Gamma', G')$ is finite.*

**Corollary (2)**

*If $G$ is unipotently connected, then there exists a finite index subgroup $\text{Aut}^\circ(\Gamma)$ of $\text{Aut}(\Gamma)$ such that every element of $\text{Aut}^\circ(\Gamma)$ extends to an automorphism of $G$. Indeed, in Corollary (2) one may take

$$\text{Aut}^\circ(\Gamma) = \text{C}_{\text{Aut}(\Gamma)}(\Gamma/\text{Fitt}(\Gamma))$$
Let $\Gamma \leq G$ be a Zariski-dense lattice.

**Definition**

$\Gamma$ is called **strongly rigid** if every isomorphism $\varphi : \Gamma \rightarrow \Gamma'$ where $\Gamma'$ is a Zariski-dense lattice in some $G'$ extends to an isomorphism

$$\hat{\varphi} : G \rightarrow G'.$$

Define the **structure set** for Zariski-dense embeddings of $\Gamma$ as

$$S^Z(\Gamma) = \{ \varphi : \Gamma \hookrightarrow G' \mid \varphi(\Gamma) \text{ is a Zariski-dense lattice in } G' \}/\sim$$
Theorem (B)

The structure set $S^Z(\Gamma)$ is either countably infinite or it consists of a single element. The structure set consists of a single element if and only if $\Gamma$ is a lattice in a solvable Lie group of real type.

Corollary (3)

Let $G$ be unipotently connected. Then $\Gamma$ is either strongly rigid or there exist countably infinite pairwise non-isomorphic simply connected (and also unipotently connected) solvable Lie groups which contain $\Gamma$ as a Zariski-dense lattice.
$G(\Gamma) = \{ G \leq A(\Gamma)_\mathbb{R} \mid G \text{ simply connected, solvable Lie subgroup and } \Gamma \text{ a (Zariski-dense) lattice in } G \}$

For every $\varphi : \Gamma \to G$, exists a unique extension $\Phi : A_\Gamma \to A_G$.

**Proposition**

*The structure map*

$$\epsilon : S^Z(\Gamma) \longrightarrow G(\Gamma) , \quad [\varphi]_{S^Z(\Gamma)} \mapsto \Phi^{-1}(G) .$$

is a bijection.