Ergodicity of generalised Patterson-Sullivan measures in Higher Rank Symmetric Spaces

Gabriele Link

Abstract

Let $X = G/K$ be a higher rank symmetric space of noncompact type and $\Gamma \subset G$ a discrete Zariski dense group. In a previous article, we constructed for each G-invariant subset of the regular limit set of Γ a family of measures, the so-called $(b, \Gamma{\cdot}\xi)$-densities. Our main result here states that these densities are Γ-ergodic with respect to an important subset of the limit set which we choose to call the "ray limit set". In the particular case of uniform lattices and products of convex cocompact groups acting on the product of rank one symmetric spaces every limit point belongs to the ray limit set, hence our result is most powerful for these examples. For nonuniform lattices, however, it is a priori not clear whether the ray limit set has positive measure with respect to a $(b, \Gamma{\cdot}\xi)$-density. Using a counting theorem of Eskin and McMullen, we are able to prove that the ray limit set has full measure in each G-invariant subset of the limit set.

1 Introduction

In 1976, S. J. Patterson ([P]) constructed a family of equivariant measures supported on the limit set of a Fuchsian group. These so-called conformal densities provide a powerful tool to relate properties of the geometric limit set to the ergodic theory of the geodesic flow of the quotient manifold. Subsequently, the theory was extended to real hyperbolic spaces by D. Sullivan ([S]) and to manifolds of variable pinched negative curvature by C. B. Yue ([Y]). However, there were certain gaps in the proofs of Yue which were filled only recently by T. Roblin ([R1], [R2]). For example, the proof of ergodicity of the densities relies largely on the existence of density points which is trivial in the context of real hyperbolic spaces, but not for manifolds of variable curvature.

In this paper we will treat the case of a globally symmetric space X of noncompact type and rank greater than one. Let $G = \mathrm{Isom}^o(X)$ be the connected component of the identity and $\Gamma \subset G$ a discrete group. We denote ∂X the geometric boundary of X endowed with the cone topology and fix $o \in X$. The geometric limit set of Γ is defined by $L_\Gamma := \overline{\Gamma{\cdot}o} \cap \partial X$.

Conformal densities in higher rank symmetric spaces were first constructed by P. Albuquerque in [A]. He also showed that for Zariski dense discrete groups $\Gamma \subset G$, the support of a conformal density is contained in a proper G-invariant subset of the geometric boundary. In order to measure the size of the limit set in the remaining G-invariant subsets of ∂X, we constructed in a previous article ([Li]) a class of generalised Patterson-Sullivan measures which we called $(b, \Gamma\cdot\xi)$-densities.

However, even in the particular case of conformal densities treated by P. Albuquerque in [A], the question of Γ-ergodicity remained open: Albuquerque's proof contained the same gap as Yue's, since the existence of density points is far from being obvious even in the case of uniform lattices.

For Schottky groups, ergodicity and uniqueness of the generalised Patterson-Sullivan measures have been proved by J. F. Quint([Qu]) using the thermodynamic formalism. Since this machinery is not available in the general case, we are going to adapt an idea of T. Roblin ([R1], [R2]) in order to prove the existence of density points for the following important subset of the limit set:

Definition 1.1 *A point $\xi \in \partial X$ is called a* ray limit point *of Γ if there exists a sequence $(\gamma_j) \subset \Gamma$ such that $\gamma_j o$ remains at bounded distance of the geodesic ray $\sigma_{o,\xi}$. The set of ray limit points is called the* ray limit set *and will be denoted L_Γ^{ray}.*

This set has been investigated for example by T. Hattori in [Ha] under the name of "conical limit set". We remark that for uniform lattices the limit set consists entirely of ray limit points. The same is true for products of convex cocompact groups acting on the Riemannian product of rank one symmetric spaces.

The existence of density points for the ray limit set together with a standard argument originally due to D. Sullivan ([S]) which was already used by P. Albuquerque ([Al]) in the context of higher rank symmetric spaces then leads to

Theorem 1. *Let $\Gamma \subset G$ be a Zariski dense discrete group, $\xi \in \partial X^{reg}$, and μ a $(b, \Gamma\cdot\xi)$-density. Then for every measurable Γ-invariant subset $A \subseteq L_\Gamma^{ray}$ either $\mu_o(A) = 0$ or $\mu_o(A) = \mu_o(\partial X)$.*

For simplicity, we restrict ourselves here as in [Al], [Li] to Zariski dense discrete groups $\Gamma \subset G$. Actually, Theorem 1 holds for a larger class of "strongly nonelementary" groups as defined in section 5.4 of [L].

If the limit set of Γ is strictly larger than the ray limit set, it is imaginable that $\mu_o(L_\Gamma^{ray}) = 0$ even for a nontrivial $(b, \Gamma\cdot\xi)$-density μ. The following result excludes this possibility in the case of irreducible nonuniform lattices.

Theorem 2. *Let $\Gamma \subset G$ be an irreducible lattice, $\xi \in \partial X^{reg}$, and μ the $(b, \Gamma\cdot\xi)$-density constructed in [Li]. Then $\mu_o(L_\Gamma^{ray} \cap G\cdot\xi) = \mu_o(\partial X)$.*

The proof of this theorem is based on a counting theorem of A. Eskin and C. McMullen ([EM]) and volume estimates in $X = G/K$ (compare [A, Appendix]).

The paper is organised as follows: In section 2 we recall some basic facts concerning Riemannian symmetric spaces of noncompact type and their geometric boundary. Section 3 is devoted to the proof of Theorem 1. We define $(b, \Gamma\cdot\xi)$-densities and prove a shadow lemma (Theorem 3.5) adapted to the ray limit set.

Extending an idea of T. Roblin ([R2]) we show that there exist density points for the ray limit set and finally prove Theorem 1. In section 4 we give a criterium which implies that $\mu_o(L_\Gamma^{ray})$ is positive and study the case of nonuniform lattices.

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2 Preliminaries

In this section we recall some well-known facts about symmetric spaces of noncompact type (see also [H1], [BGS], [E]) and introduce some notation for the sequel.

Let X be a simply connected symmetric space of noncompact type with base point $o \in X$, $G = \mathrm{Isom}^o(X)$, and K the isotropy subgroup of o in G. It is well-known that G is a semisimple Lie group with trivial centre and no compact factors, $K \subset G$ a maximal compact subgroup, and X is isometric to the homogeneous space G/K endowed with an appropriate G-invariant Riemannian metric. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K. The geodesic symmetry in X induces a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that the tangent space T_oX is identified with the subspace $\mathfrak{p} \subset \mathfrak{g}$.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and $\mathfrak{a}_1 \subset \mathfrak{a}$ the set of unit vectors. The rank r of X is defined as $r = \dim \mathfrak{a}$, and we denote by Σ the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$. The choice of an open Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ determines a set of positive roots $\Sigma^+ \subset \Sigma$. Fixing a set of simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ of Σ^+ we obtain a basis $\{H_1, H_2, \ldots, H_r\}$ of $\mathfrak{a}$ with the property $H_i \in \mathfrak{a}_1$, $\alpha_i(H_i) > 0$ and $\alpha_j(H_i) = 0$ for all $i, j \in \{1, 2, \ldots, r\}$, $i \neq j$. $H_1, H_2, \ldots, H_r$ are called the maximal singular directions.

If $\overline{\mathfrak{a}^+}$ denotes the closure of $\mathfrak{a}^+$, then $G = Ke^{\overline{\mathfrak{a}^+}}K$ is a Cartan decomposition of G, and every point $x \in X$ can be written as $x = ke^H o$ with $k \in K$ and a unique $H \in \overline{\mathfrak{a}^+}$. We call k an angular projection and H the Cartan projection of x.

The geometric boundary ∂X of X is defined as the set of equivalence classes of asymptotic geodesic rays endowed with the cone topology. For $k \in K$ and $H \in \overline{\mathfrak{a}_1^+}$, we denote by (k, H) the unique class in ∂X which contains the geodesic ray $\sigma(t) = ke^{Ht}o$, $t > 0$. Again k is called an angular projection and H the Cartan projection of (k, H).

The action of an isometry $g \in G$ extends naturally to a homeomorphism on the geometric boundary ∂X. If $\xi = (k, H) \in \partial X$, then every point in the G-orbit $G\cdot\xi = K\cdot\xi$ of ξ has the same Cartan projection $H \in \overline{\mathfrak{a}_1^+}$. In particular, G acts transitively on ∂X if and only if $r = \mathrm{rank}(X) = 1$.

We define the maximal singular boundary components $\partial X^1, \partial X^2, \ldots, \partial X^r$ by $\partial X^i := \{(k, H_i) |\ k \in K\}$ for $1 \le i \le r$. Here $H_1, H_2, \ldots, H_r \in \overline{\mathfrak{a}_1^+}$ are the maximal singular directions described above. The regular boundary $\partial X^{reg} \subseteq \partial X$ is defined as the set of classes with Cartan projection $H \in \mathfrak{a}_1^+$.

Given regular boundary points $\xi, \zeta \in \partial X^{reg}$, the big Bruhat cell in $G{\cdot}\xi$ corresponding to ζ is the dense and open subset of $G{\cdot}\xi$ defined by

$$\{\eta \in G{\cdot}\xi \mid \exists \text{ geodesic } \sigma \text{ such that } \sigma(\infty) = \eta \text{ and } \sigma(-\infty) \text{ has the same angular projection as } \zeta\}\,.$$

Let $\overline{X} := X \cup \partial X$. For $x \in X$ and $z \in \overline{X}$ we denote by $\sigma_{x,z}$ the unique unit speed geodesic emanating from x which contains z. Let $x, y \in X$, $\xi \in \partial X$ and σ a geodesic ray in the class of ξ. The number

$$\mathcal{B}_\xi(x,y) := \lim_{s\to\infty} \big(d(x,\sigma(s)) - d(y,\sigma(s))\big)$$

is independent of the chosen ray σ, and the function $\mathcal{B}_\xi(\cdot, y) : X \to \mathbb{R}$ is called the Busemann function centred at ξ based at y.

We will further need the following (possibly nonsymmetric) G-invariant pseudo distances on X (see [Li, section 2.3]): For $1 \le i \le r$, $x, y \in X$ we put

$$d_i(x,y) := \sup_{\xi \in \partial X^i} \mathcal{B}_\xi(x,y)\,.$$

Notice that if $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathfrak{p} \cong T_o X$, and $H \in \overline{\mathfrak{a}^+}$ the Cartan projection of y, we have $d_i(o,y) = \langle H_i, H\rangle$.

3 $(b, \Gamma{\cdot}\xi)$-densities and shadows

For this section we let $\Gamma \subset G$ be a Zariski dense discrete group of isometries of a higher rank symmetric space. Fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ of G and denote by M the centraliser of $\mathfrak{a}$ in K. It is well-known (see for example [L, Theorem 5.15], [Be]) that the regular geometric limit set splits as a product $K_\Gamma \times (P_\Gamma \cap \mathfrak{a}_1^+)$, where $K_\Gamma = \{kM \in K/M \mid \exists\ H \in \mathfrak{a}_1^+ \text{ such that } (k, H) \in L_\Gamma\}\,,$ and $P_\Gamma \subseteq \overline{\mathfrak{a}_1^+}$ is the set of Cartan projections of limit points. In particular, for any $H \in P_\Gamma \cap \mathfrak{a}_1^+$, $\xi = (\mathrm{id}, H) \in \partial X^{reg}$, the set $L_\Gamma \cap G{\cdot}\xi$ is a Γ-invariant subset of the limit set isomorphic to $K_\Gamma \times \{H\}$. In order to measure the limit set in each subset $G{\cdot}\xi$ of the regular boundary ∂X^{reg}, the following families of measures are useful:

Definition 3.1 *Let $\mathcal{M}^+(\partial X)$ denote the cone of positive finite Borel measures on ∂X, $\xi \in \partial X^{reg}$ and $b = (b^1, b^2, \dots b^r) \in \mathbb{R}^r$. A $(b, \Gamma{\cdot}\xi)$-density is a continuous map*

$$\begin{array}{cccc} \mu: & X & \to & \mathcal{M}^+(\partial X) \\ & x & \mapsto & \mu_x \end{array}$$

with the properties

$$(i) \qquad supp(\mu_o) \subseteq L_\Gamma \cap G{\cdot}\xi\,,$$
$$(ii) \qquad \gamma * \mu_x = \mu_{\gamma^{-1}x} \quad \textit{for any } \gamma \in \Gamma\,,\ x \in X\,,$$
$$(iii) \qquad \frac{d\mu_x}{d\mu_o}(\eta) = e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(o,x)} \quad \textit{for any } x \in X\,,\ \eta \in supp(\mu_o)\,.$$

Here $\eta_i \in \partial X^i$ denotes the unique point in the i-th maximal singular boundary component which is contained in the closure of the Weyl chamber at infinity determined by $\eta \in \partial X^{reg}$.

Remark. If r =rank$(X) = 1$ then for any $\xi \in \partial X$ we have $L_\Gamma \cap G{\cdot}\xi = L_\Gamma$, and the above definition for a $(b, \Gamma{\cdot}\xi)$-density is exactly the definition of a b-dimensional conformal density.

In section 4.1 of [Li] we showed how to construct for any $\xi \in \partial X^{reg}$ with Cartan projection $H_\xi \in P_\Gamma$ a $(b, \Gamma{\cdot}\xi)$-density. For a unique subset $G{\cdot}\xi_* \subset \partial X^{reg}$ this $(b, \Gamma{\cdot}\xi_*)$-density is exactly the $\delta(\Gamma)$-dimensional conformal density constructed by P. Albuquerque in [Al].

In higher rank symmetric spaces, the definition of radial or conical limit points is ambigous. In [L], [Li] we were mainly working with the following definition, which was motivated by P. Albuquerque ([Al]):

Definition 3.2 *A point $\xi \in \partial X$ is called a* radial limit point *of Γ if there exists a sequence $(\gamma_j) \subset \Gamma$ such that $\gamma_j o$ converges to ξ and remains at bounded distance of the union of closed Weyl chambers with apex o containing the geodesic ray $\sigma_{o,\xi}$. The set of radial limit points is denoted by L_Γ^{rad}.*

Notice that this definition is in general weaker than the one given in the introduction, i.e. $L_\Gamma^{ray} \subseteq L_\Gamma^{rad}$. However, in rank one symmetric spaces, the two definitions coincide.

Following P. Albuquerque ([Al, Definition 3.2]), we introduce the notion of Furstenberg shadow $S(x : B) \subseteq \partial X$ for $B \subset X$, $x \in X \setminus B$, which is defined as the set of those points in the geometric boundary which belong to the closure of all Weyl chambers with apex x intersecting B nontrivially. Let $B_z(r) \subset X$ denote the ball of radius r centred at $z \in X$. For $x, y \in X$, $\xi \in \partial X$ we further put $\angle_x(y, G{\cdot}\xi) := \inf\{\angle_x(y,\zeta) \,|\, \zeta \in G{\cdot}\xi\}$. Since

$$L_\Gamma^{rad} \cap G{\cdot}\xi = \bigcup_{c>0} \bigcap_{R>c} \bigcap_{\varphi>0} \bigcup_{\substack{\gamma\in\Gamma \\ d(o,\gamma o)>R \\ \angle_o(\gamma o, G\xi)<\varphi}} S(o : B_{\gamma o}(c)) \cap G{\cdot}\xi\,,$$

the following shadow lemma proved in section 4.3 of [L] turns out to be useful when dealing with the radial limit set:

Theorem 3.3 (Shadow Lemma for Furstenberg shadows) *Let $\Gamma \subset G$ be a Zariski dense discrete group, $\xi \in \partial X^{reg}$, and μ a $(b, \Gamma{\cdot}\xi)$-density. Then there exists a constant $c' > 0$ such that for any $c > c'$ and for every Γ-invariant measurable set $F \subseteq \partial X$ with $\mu_o(F) > 0$ there exists a constant $D(c) > 1$ with the property*

$$\frac{1}{D(c)} e^{-\sum_{i=1}^r b^i d_i(o,\gamma o)} \le \mu_o\left(S(o : B_{\gamma o}(c)) \cap F\right) \le D(c) e^{-\sum_{i=1}^r b^i d_i(o,\gamma o)}$$

for $\gamma o \in X \setminus B_o(c)$ such that $S(o : B_{\gamma o}(c)) \cap F \neq \emptyset$.

However, we are not able to prove the existence of density points with respect to Furstenberg shadows, because the shape of a set $S(o : B_{\gamma o}(c)) \cap G{\cdot}\xi$ depends strongly on the Cartan projection of γo. The shadows at infinity $\mathrm{sh}_x(B) \subseteq \partial X$ for $B \subset X$, $x \in X \setminus B$ defined by

$$\mathrm{sh}_x(B) := \{\eta \in \partial X \,|\, \sigma_{x,\eta} \cap B \neq \emptyset\},$$

are easier to deal with. For this reason we choose to work here with the ray limit set which satisfies

$$L_\Gamma^{ray} \cap G{\cdot}\xi = \bigcup_{c>0} \bigcap_{R>c} \bigcup_{\substack{\gamma\in\Gamma \\ d(o,\gamma o)>R \\ d(\gamma o,\sigma_{o,G\xi})<c/3}} \mathrm{sh}_o(B_{\gamma o}(c)), \qquad (1)$$

where $\sigma_{x,G\xi} \subset \overline{X}$ denotes the set of all geodesic rays $\sigma_{x,\zeta}$ with $\zeta \in G\cdot\xi$. In order to deal with ray limit points, we have to adapt the shadow lemma to the smaller shadows at infinity. This can be achieved by giving a relation between Furstenberg shadows and shadows at infinity. From the above definitions it is clear that $\mathrm{sh}_x(B) \subseteq S(x : B)$ for any subset $B \subset X$, $x \in X \setminus B$. Conversely, we have the following

Lemma 3.4 *Let $c > 0$, $\xi \in \partial X$ and $z \in X \setminus B_o(3c)$ such that $d(z, \sigma_{o,G\xi}) < c$. Then*

$$S(o : B_z(c)) \cap G{\cdot}\xi \subseteq sh_o(B_z(3c)).$$

Proof. Let $H_\xi \in \overline{\mathfrak{a}_1^+}$, $H_z \in \overline{\mathfrak{a}^+}$ be the Cartan projections of ξ and z. Then $d(z, \sigma_{o,G\xi}) = d(z, K\cdot\sigma_{o,\xi}) = d(e^{H_z}o, \sigma_{o,\xi}) < c$, i.e. there exists $t > 0$ such that $d(e^{H_z}o, e^{H_\xi t}o) = d(e^{H_z}o, \sigma_{o,\xi}(t)) = d(e^{H_z}o, \sigma_{o,\xi}) < c$.

If $\eta \in S(o : B_z(c)) \cap G\cdot\xi$, then η possesses an angular projection $k \in K$ such that $d(z, ke^{\overline{\mathfrak{a}^+}}o) < c$. This implies the existence of $H \in \overline{\mathfrak{a}^+}$ such that $d(e^{H_z}o, e^H o) \le d(z, ke^H o) < c$. We conclude

$$\begin{aligned} d(z,\sigma_{o,\eta}) &\le d(z, k\sigma_{o,\xi}(t)) \le d(z, ke^H o) + d(ke^H o, ke^{H_z}o) + d(ke^{H_z}o, ke^{H_\xi t}o) \\ &\le c + d(e^H o, e^{H_z}o) + d(e^{H_z}o, e^{H_\xi t}o) < 3c. \end{aligned}$$

□

Theorem 3.5 (Shadow Lemma for shadows at infinity) *Let $\Gamma \subset G$ be a Zariski dense discrete group, $\xi \in \partial X^{reg}$, and μ a $(b, \Gamma{\cdot}\xi)$-density. Then there exists a constant $c_0 > 0$ such that for any $c > c_0$ and for every Γ-invariant measurable set $F \subseteq G{\cdot}\xi$ with $\mu_o(F) > 0$ there exists a constant $D = D(c) > 1$ with the property*

$$\frac{1}{D} e^{-\sum_{i=1}^r b^i d_i(o,\gamma o)} \le \mu_o\left(sh_o(B_{\gamma o}(c)) \cap F\right) \le D e^{-\sum_{i=1}^r b^i d_i(o,\gamma o)}$$

if $\gamma o \in X \setminus B_o(c)$ satisfies $d(\gamma o, \sigma_{o,G\xi}) < c/3$ and $sh_o(B_{\gamma o}(c)) \cap F \neq \emptyset$.

Proof. If $\gamma o \in X \setminus B_o(c)$ satisfies $\mathrm{sh}_o(B_{\gamma o}(c)) \cap F \neq \emptyset$, then $S(o : B_{\gamma o}(c)) \cap F \neq \emptyset$. Furthermore, if $d(\gamma o, \sigma_{o,G\xi}) < c/3$, then by the previous lemma $\mathrm{sh}_o(B_{\gamma o}(c)) \supseteq S(o : B_{\gamma o}(c/3)) \cap G{\cdot}\xi$. Hence for $c > 3c'$ with c' as in Theorem 3.3 we obtain

$$\begin{aligned} \frac{1}{D(c/3)} e^{-\sum_{i=1}^r b^i d_i(o,\gamma o)} &\le \mu_o\left(S(o : B_{\gamma o}(c/3)) \cap F\right) \le \mu_o\left(\mathrm{sh}_o(B_{\gamma o}(c)) \cap F\right) \\ &\le \mu_o\left(S(o : B_{\gamma o}(c)) \cap F\right) \le D(c) e^{-\sum_{i=1}^r b^i d_i(o,\gamma o)} . \end{aligned}$$

Hence $c_0 = 3c'$ and $D = D(c)$ do the trick. □

3.1 Density points and ergodicity

For $c > 0$ we denote

$$L_\Gamma(G{\cdot}\xi, c) := \bigcap_{R>c} \bigcup_{\substack{\gamma\in\Gamma \\ d(o,\gamma o)>R \\ d(\gamma o, \sigma_{o,G\xi})<c/3}} \mathrm{sh}_o(B_{\gamma o}(c)) .$$

Then by definition of the ray limit set (1) we have $L_\Gamma^{ray} \cap G{\cdot}\xi \subseteq \bigcup_{c>0} L_\Gamma(G{\cdot}\xi, c)$. For $\gamma \in \Gamma$ with $d(o, \gamma o) > c$ we further put

$$S^c(\gamma) := \mathrm{sh}_o(B_{\gamma o}(c)) \subseteq \partial X .$$

Lemma 3.6 *Suppose μ is a $(b, \Gamma{\cdot}\xi)$-density and $\Phi : \partial X \to \mathbb{R}^+$ a positive bounded Borel function. If $c > c_0$ with c_0 as in Theorem 3.5, then for μ_o-almost every point $\eta \in L_\Gamma(G{\cdot}\xi, c)$ we have*

$$\lim_{\substack{d(o,\gamma o)\to\infty \\ \eta\in S^{c/3}(\gamma)}} \frac{1}{\mu_o(S^c(\gamma))} \int_{S^c(\gamma)} \Phi d\mu_o = \Phi(\eta) .$$

Proof. Fix $c > c_0$ and introduce the maximal function $\Phi^* : \partial X \to \mathbb{R}^+$ defined by $\Phi^*(\eta) = 0$ for $\eta \notin L_\Gamma(G{\cdot}\xi, c)$, and

$$\Phi^*(\eta) = \limsup_{R\to\infty} \frac{1}{\mu_o(S^c(\gamma))} \int_{S^c(\gamma)} \Phi d\mu_o \quad \text{for } \eta \in L_\Gamma(G{\cdot}\xi, c) ,$$

where the limit superior is taken over those $\gamma \in \Gamma$ with $d(o, \gamma o) > R$ and $\eta \in S^{c/3}(\gamma)$.

If Φ is continuous on $L_\Gamma(G{\cdot}\xi, c)$, there is nothing to prove. If not, there exists a sequence (Φ_n) of continuous functions on $L_\Gamma(G{\cdot}\xi, c)$ which converge μ_o-almost surely to Φ and such that $\int_{\partial X} |\Phi_n - \Phi| d\mu_o \to 0$. Then for any point $\eta \in L_\Gamma(G{\cdot}\xi, c)$ and $\gamma \in \Gamma$ such that $\eta \in S^{c/3}(\gamma)$

$$\begin{aligned} \Big|\frac{1}{\mu_o(S^c(\gamma))} \int_{S^c(\gamma)} \Phi d\mu_o - \Phi(\eta)\Big| &\le |\Phi - \Phi_n|^*(\eta) + |\Phi_n(\eta) - \Phi(\eta)| \\ &\quad + \Big|\frac{1}{\mu_o(S^c(\gamma))} \int_{S^c(\gamma)} \Phi_n d\mu_o - \Phi_n(\eta)\Big| \,. \end{aligned}$$

The continuity of the function Φ_n then implies

$$\limsup_{\substack{d(o,\gamma o)\to\infty \\ \eta \in S^{c/3}(\gamma)}} \Big|\frac{1}{\mu_o(S^c(\gamma))} \int_{S^c(\gamma)} \Phi d\mu_o - \Phi(\eta)\Big| \le |\Phi - \Phi_n|^*(\eta) + |\Phi_n(\eta) - \Phi(\eta)| \,.$$

As $n \to \infty$ the last term tends to zero for μ_o-almost every $\eta \in L_\Gamma(G\cdot\xi, c)$. It therefore suffices to prove the following: There exists a constant $M_0 > 0$ such that for any positive bounded Borel function $\Psi : \partial X \to \mathbb{R}^+$ and every $\varepsilon > 0$

$$\mu_o(\{\Psi^* > \varepsilon\}) \le \frac{M_0}{\varepsilon} \int_{\partial X} \Psi d\mu_o \,. \tag{2}$$

Let $\varepsilon > 0$ arbitrary. Since $c > c_0$, Theorem 3.5 yields the existence of a constant $D > 0$ such that for any $\gamma \in \Gamma$ with $d(o, \gamma o) > c$, $d(\gamma o, \sigma_{o,G\xi}) < c/3$ we have

$$e^{-\sum_{i=1}^r b^i d_i(o,\gamma o)} \le D\mu_o(S^c(\gamma))\,, \quad \text{and} \quad \mu_o(S^{3c}(\gamma)) \le D e^{-\sum_{i=1}^r b^i d_i(o,\gamma o)} \,. \tag{3}$$

Let Γ' be a subset of elements of Γ such that $\{\Psi^* > \varepsilon\} \subset \cup_{\gamma\in\Gamma'} S^c(\gamma)$ and any $\gamma \in \Gamma'$ satisfies $d(o, \gamma o) \ge c$, $d(\gamma o, \sigma_{o,\zeta}) < c/3$ for some $\zeta \in L_\Gamma(G \cdot \xi, c)$, and $\int_{S^c(\gamma)} \Psi d\mu_o > \frac{\varepsilon}{2}\mu_o(S^c(\gamma))$. We construct recursively a sequence (Γ_k) of subsets of Γ' in the following way: Let $\Gamma_1 := \{\gamma \in \Gamma' \mid c \le d(o, \gamma o) < c + 1\}$, and for $k \ge 2$ put

$$\begin{aligned} \Gamma_k := \{\gamma \in \Gamma' \mid c + k - 1 \le d(o, \gamma o) < c + k \quad \text{and} \quad & S^c(\gamma) \cap S^c(\varphi) = \emptyset \\ & \text{for} \;\; \varphi \in \Gamma_1 \cup \ldots \cup \Gamma_{k-1}\} \,. \end{aligned}$$

We put $\Gamma^* := \cup_{k\ge1}\Gamma_k$. If $\gamma \in \Gamma' \setminus \Gamma^*$, there exists $k \ge 2$ and $\varphi \in \Gamma_1 \cup \ldots \cup \Gamma_{k-1}$ such that $c + k - 1 \le d(o, \gamma o) < c + k$ and $S^c(\gamma) \cap S^c(\varphi) \ne \emptyset$. Hence there exists a geodesic ray emanating from o which intersects both $B_{\varphi o}(c)$ and $B_{\gamma o}(c)$. This implies $S^c(\gamma) \subseteq S^{3c}(\varphi)$ and therefore

$$\bigcup_{\gamma\in\Gamma'} S^c(\gamma) \subseteq \bigcup_{\gamma\in\Gamma^*} S^{3c}(\gamma) \,.$$

From the definition of Γ' and (3) we conclude

$$\mu_o(\{\Psi^* > \varepsilon\}) \leq \sum_{\gamma\in\Gamma'} \mu_o(S^c(\gamma)) \leq \sum_{\gamma\in\Gamma^*} \mu_o(S^{3c}(\gamma)) \leq D^2 \sum_{\gamma\in\Gamma^*} \mu_o(S^c(\gamma))\,.$$

Furthermore, if $\gamma, \varphi \in \Gamma^*$ such that $S^c(\gamma) \cap S^c(\varphi) \neq \emptyset$, then by construction of Γ^* both γo and φo belong to a same annulus of the form $\{x \in X \mid c+k-1 \leq d(o,x) < c+k\}$, i.e. $d(\gamma o, \varphi o) \leq 2c+1$. Hence there exists a constant $A > 0$ such that

$$\sum_{\gamma\in\Gamma^*} \mu_o(S^c(\gamma)) \leq A\mu_o\big(\bigcup_{\gamma\in\Gamma^*} S^c(\gamma)\big)\,.$$

We finally obtain

$$\frac{\varepsilon}{2} \sum_{\gamma\in\Gamma^*} \mu_o(S^c(\gamma)) \leq \sum_{\gamma\in\Gamma^*} \int_{S^c(\gamma)} \Psi d\mu_o \leq A \int_{\partial X} \Psi d\mu_o$$

which implies (2) with $M_0 = 2AD^2$. □

The following statement is due to Y. Benoist and will be essential in the proof of Theorem 3.10. It is cited and proved in [Al], Lemma 5.5.

Lemma 3.7 *Let G be a connected algebraic group, $\Gamma \subseteq G$ a Zariski dense discrete subgroup, V an algebraic irreducible homogeneous G-space, Y a proper algebraic subvariety of V and m a probability measure on V. Then, for any $\varepsilon > 0$, the set $\Gamma_\varepsilon = \{\gamma \in \Gamma \mid m(\gamma Y) < \varepsilon\}$ is Zariski dense.*

For the reader's convenience we recall Lemma 7.3 of [L] (see also [Al, Lemma 3.5]):

Lemma 3.8 *Fix $\xi \in \partial X$. Then for any $\varepsilon > 0$ there exists $c' > 0$ such that for all $c > c'$ and $z \in X \setminus B_o(c)$ the set $G\cdot\xi \setminus (S(z : B_o(c)) \cap G\cdot\xi)$ is contained in the ε-neighbourhood of the complement of the big Bruhat cell in $G\cdot\xi$ corresponding to $\sigma_{z,o}(\infty)$.*

We will further need the following relation between $\mathcal{B}_{\eta_i}(o,z)$ and $d_i(o,z)$ for certain points $z \in X$, $\eta_i \in \partial X^i$, $1 \leq i \leq r$ (see [Al, Lemma 3.6], [L, Corollary 7.5]):

Lemma 3.9 *Let $c > 0$, $z \in X$ with $d(o,z) > c$. Then*

$$\forall\, \eta_i \in \partial X^i \cap S(o : B_z(c)) : \qquad 0 \leq d_i(o,z) - \mathcal{B}_{\eta_i}(o,z) < 2c\,. \tag{4}$$

Theorem 3.10 *Let $\Gamma \subset G$ be a Zariski dense discrete group, $\xi \in \partial X^{reg}$ and μ a $(b, \Gamma\cdot\xi)$-density. Then for every Γ-invariant subset $A \subseteq L_\Gamma^{ray}$ either $\mu_o(A) = 0$ or $\mu_o(A) = \mu_o(\partial X)$.*

Proof. Let $A \subseteq L_\Gamma^{ray}$ be a Γ-invariant subset with $\mu_o(A) > 0$ and $C > c_0$ such that $\mu_o(A \cap L_\Gamma(G\cdot\xi, C)) > 0$. Applying Lemma 3.6 with Φ the characteristic function of

A, for each $c > C$ there exists a point $\eta \in A \cap L_\Gamma(G\cdot\xi, c)$ and a sequence $(\gamma_j) \subset \Gamma$ such that $\eta \in S^{c/3}(\gamma_j)$ for all $j \in \mathbb{N}$ and

$$\lim_{j\to\infty} \frac{\mu_o(S^c(\gamma_j) \cap A)}{\mu_o(S^c(\gamma_j))} = 1\,.$$

Hence for each $n \in \mathbb{N}$, $n > C$, we may choose $\eta_n \in A \cap L_\Gamma(G\cdot\xi, n)$, and $\gamma_n \in \Gamma$ such that $d(o, \gamma_n o) > n$, $\eta_n \in S^{n/3}(\gamma_n)$ and

$$\frac{\mu_o\big(S^n(\gamma_n) \setminus (A \cap S^n(\gamma_n))\big)}{\mu_o(S^n(\gamma_n))} < e^{-3n\|b\|_1}\,. \tag{5}$$

Let $\varepsilon > 0$ arbitrarily small. According to Lemma 3.8 there exists $c' > C$ such that for all $c > c'$ and any $z \in X \setminus B_o(c)$ the set $G\cdot\xi \setminus (S(z : B_o(c)) \cap G\cdot\xi)$ is contained in the ε-neighbourhood of the complement of a big Bruhat cell in $G\cdot\xi$ which depends only on o and z. In particular, for all $n \in \mathbb{N}$, $n > c'$, the set $G\cdot\xi \setminus (S(\gamma_n^{-1}o : B_o(n)) \cap G\cdot\xi)$ is contained in the ε-neigbourhood of the complement of the big Bruhat cell $Y_n \subset G\cdot\xi$ corresponding to $\sigma_{\gamma_n^{-1}o,o}(\infty)$. The sequence $(Y_n) \subset G\cdot\xi$ possesses an accumulation point $Y \subset G\cdot\xi$ with respect to the Hausdorff topology which is the complement of a big Bruhat cell in $G\cdot\xi$. Since Y is a proper algebraic subvariety of $G\cdot\xi$, Lemma 3.7 implies the existence of $\gamma \in \Gamma$ such that $\mu_o(\gamma Y) < \varepsilon$.

Let $U \subset G\cdot\xi$ be an open neighbourhood of γY such that $\mu_o(U) < 2\varepsilon$, and $\varepsilon' > 0$ such that the ε'-neighbourhood of γY is contained in U. Using again Lemma 3.8 there exists $c' > 3d(o, \gamma o) + 3C$ such that for all $n > c'$ the sets

$$\begin{aligned}\gamma\big(G\cdot\xi \setminus (S(\gamma_n^{-1}o : B_o(c'/3)) \cap G\cdot\xi)\big) &= G\cdot\xi \setminus (S(\gamma\gamma_n^{-1}o : B_{\gamma o}(c'/3)) \cap G\cdot\xi)\\ &\subseteq\ G\cdot\xi \setminus (S(\gamma\gamma_n^{-1}o : B_o(c'/3 - d(o,\gamma o))) \cap G\cdot\xi)\end{aligned}$$

are contained in an $\varepsilon'/2$-neighbourhood of γY_n. By Lemma 3.4, the same is true for the smaller sets $G\cdot\xi \setminus (\mathrm{sh}_{\gamma\gamma_n^{-1}o}(B_{\gamma o}(c')) \cap G\cdot\xi)$.

Let $(n_j) \subset \mathbb{N}$ be a sequence such that γY_{n_j} is contained in an $\varepsilon'/2$-neighbourhood of γY for all $j \in \mathbb{N}$. Then for all $j \in \mathbb{N}$ with $n_j > c'$ we have

$$\begin{aligned}\mu_o\big(\mathrm{sh}_{\gamma\gamma_{n_j}^{-1}o}(B_{\gamma o}(n_j))\big) \ &\geq\ \mu_o\big(\mathrm{sh}_{\gamma\gamma_{n_j}^{-1}o}(B_{\gamma o}(c'))\big) = \mu_o(\partial X)\\ &\qquad - \mu_o\big(\partial X \setminus \mathrm{sh}_{\gamma\gamma_{n_j}^{-1}o}(B_{\gamma o}(c'))\big) > \mu_o(\partial X) - 2\varepsilon\,.\end{aligned} \tag{6}$$

Put $S_j := S^{n_j}(\gamma_{n_j}) = \mathrm{sh}_o(B_{\gamma_{n_j}o}(n_j))$, and $A_j := A \cap S_j$. We compute

$$\begin{aligned}\frac{\mu_o(\mathrm{sh}_{\gamma\gamma_{n_j}^{-1}o}(B_{\gamma o}(n_j)) \cap A)}{\mu_o\big(\mathrm{sh}_{\gamma\gamma_{n_j}^{-1}o}(B_{\gamma o}(n_j))\big)} &= 1 - \frac{\mu_o(\gamma\gamma_{n_j}^{-1}(S_j \setminus A_j))}{\mu_o(\gamma\gamma_{n_j}^{-1}S_j)} = 1 - \frac{\mu_{\gamma_{n_j}\gamma^{-1}o}(S_j \setminus A_j)}{\mu_{\gamma_{n_j}\gamma^{-1}o}(S_j)}\\ =\quad 1 - \frac{\int_{S_j\setminus A_j} d\mu_{\gamma_{n_j}\gamma^{-1}o}(\eta)}{\int_{S_j} d\mu_{\gamma_{n_j}\gamma^{-1}o}(\eta)} &= 1 - \frac{\int_{S_j\setminus A_j} e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(o,\gamma_{n_j}\gamma^{-1}o)} d\mu_o(\eta)}{\int_{S_j} e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(o,\gamma_{n_j}\gamma^{-1}o)} d\mu_o(\eta)}\\ \overset{(4)}{\geq}\quad 1 - \frac{e^{\sum_{i=1}^r b^i d_i(o,\gamma_{n_j}\gamma^{-1}o)}\mu_o(S_j \setminus A_j)}{e^{\sum_{i=1}^r b^i d_i(o,\gamma_{n_j}\gamma^{-1}o) - 2n_j\|b\|_1}\mu_o(S_j)} &\overset{(5)}{\geq} 1 - e^{-n_j\|b\|_1} > 1 - \varepsilon\end{aligned} \tag{7}$$

for j sufficiently large. We conclude

$$\begin{aligned}\mu_o(A) &\geq \mu_o(\mathrm{sh}_{\gamma\gamma_{n_j}^{-1}o}(B_{\gamma o}(n_j)) \cap A) \overset{(7)}{>} (1-\varepsilon)\mu_o(\mathrm{sh}_{\gamma\gamma_{n_j}^{-1}o}(B_{\gamma o}(n_j))) \\ &\overset{(6)}{>} (1-\varepsilon)\,(\mu_o(\partial X) - 2\varepsilon)\ .\end{aligned}$$

Letting $\varepsilon \searrow 0$ we obtain $\mu_o(A) \geq \mu_o(\partial X)$. □

4 Groups with large ray limit set

Fix $\xi \in \partial X^{reg}$ and let μ be a $(b, \Gamma\cdot\xi)$-density. We are going to give a criterium which implies that the μ_o-measure of the ray limit set is positive.

Recall that $\sigma_{o,G\xi} \subset \overline{X}$ denotes the set of geodesic rays $\sigma_{o,\zeta}$ with $\zeta \in G\cdot\xi$. For $c > 0$ we put

$$\Gamma(G\cdot\xi, c) := \{\gamma \in \Gamma \mid d(\gamma o, \sigma_{o,G\xi}) < c\}\,.$$

Proposition 4.1 *Fix $\xi \in \partial X$ and suppose there exist $s > 0$, $C > 0$ such that*

$$\liminf_{l\to\infty} \sum_{\substack{\gamma\in\Gamma(G\xi,C)\\ l-s\leq d(o,\gamma o)<l}} e^{-\sum_{i=1}^r b^i d_i(o,\gamma o)} > 0\,.$$

Then $\mu_o(L_\Gamma^{ray}) > 0$.

Proof. We put

$$S(l) := \sum_{\substack{\gamma\in\Gamma(G\xi,C)\\ l-s\leq d(o,\gamma o)<l}} e^{-\sum_{i=1}^r b^i d_i(o,\gamma o)}\,.$$

By Theorem 3.5 we have for $l >> s$ and $c > \max\{c_0, 3C\}$

$$S(l) \leq D \sum_{\substack{\gamma\in\Gamma(G\xi,C)\\ l-s\leq d(o,\gamma o)<l}} \mu_o\big(\mathrm{sh}_o(B_{\gamma o}(c))\big)\,.$$

Now if $\eta \in L_\Gamma^{ray}$, then η belongs to at most $M = M(s,c)$ shadows of the form $\mathrm{sh}_o(B_{\gamma o}(c))$ with $\gamma \in \Gamma(G\cdot\xi, C)$, $l - s \leq d(o,\gamma o) < l$ by discreteness of Γ. Hence for $l >> s$ we have $S(l) \leq M\ D\ \mu_o(L_\Gamma^{ray})$, in particular

$$\mu_o(L_\Gamma^{ray}) \geq \tfrac{1}{MD} \liminf_{l\to\infty} S(l) > 0\,.$$ □

4.1 The case of nonuniform lattices

For this section we let $\Gamma \subset G$ be an irreducible lattice. Fix $\xi \in \partial X^{reg}$, $s > 0$ and $C > 0$. For $l \in \mathbb{N}$, $l > s$ we are going to estimate the number

$$N_{\Gamma\xi}^C(l) := \#\{\gamma \in \Gamma(G\cdot\xi, C) \mid l - s \leq d(o,\gamma o) < l\}\,,$$

using the following particular case of a counting theorem of A. Eskin and C. McMullen ([EM]):

Proposition 4.2 (A. Eskin, C. McMullen) *Let $X = G/K$ be a Riemannian symmetric space of noncompact type and Γ an irreducible lattice in G. If $(B_n) \subset X$ is a well-rounded sequence such that* vol $B_n \to \infty$ *as* $n \to \infty$, *then*

$$\mathrm{vol}\ B_n \asymp \#\{\Gamma \cdot o \cap B_n\}$$

Here a sequence of subsets $(B_n) \subset X$ with boundaries (∂B_n) is called well-rounded if for any $\varphi > 0$ there exists a neighbourhood U of the identity in G such that

$$\mathrm{vol}\ (U \cdot \partial B_n) < \varphi \cdot \mathrm{vol}\ B_n \qquad \text{for all} \quad n \in \mathbb{N}\,.$$

We further say that two real valued functions f, g are asymptotically equivalent $f \asymp g$, if there exists a constant $D > 1$ such that

$$\frac{1}{D} f(t) \leq g(t) \leq D f(t) \quad \text{as} \qquad t \to \infty\,.$$

Put $C(l) := \{z \in X \mid l - s \leq d(o,z) < l,\ d(z, \sigma_{o,G\xi}) < C\}$. By Proposition 4.2 we have $\#\{\gamma \in \Gamma(G\cdot\xi, C) \mid l - s \leq d(o, \gamma o) < l\} \asymp$ vol $C(l)$ if $C(l)$ is a well-rounded sequence.

Lemma 4.3 *The sequence of subsets $C(l) \subset X$ is well-rounded.*

Proof. Fix $\varphi > 0$ and a Cartan decomposition $G = K e^{\overline{\mathfrak{a}^+}} K$. We have to find a neighbourhood of the identity U in G such that vol $(U \cdot \partial C(l)) < \varphi \cdot$ vol $C(l)$ holds as l tends to infinity. For $l > s$ we put

$$\mathfrak{a}(l) := \{H \in \overline{\mathfrak{a}^+} \mid l - s \leq \|H\| < l,\ \|H - H_\xi \langle H_\xi, H\rangle\| < C\}\,.$$

Then $C(l) = K e^{\mathfrak{a}(l)} o$, and for l sufficiently large we have $\alpha(H) \geq const > 0$ for any $H \in \mathfrak{a}(l)$ and all $\alpha \in \Sigma^+$. Let $\rho := \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ denote the sum of positive roots α counted with multiplicity m_α, χ_Y the characteristic function of a set Y, dk_M the K-invariant measure on K/M and dH the Lebesgue measure on $\mathfrak{a} \cong \mathbb{R}^r$. Using the integration formula ([H2], chapter 1, §5) in polar coordinates, we compute

$$\begin{aligned}
\mathrm{vol}\ C(l) &= \int_{K/M} \Big(\int_{\mathfrak{a}^+} \chi_{K e^{\mathfrak{a}(l)} o}(k e^H o) \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^{m_\alpha} dH \Big) dk_M \\
&= \int_{K/M} \chi_K(k) dk_M \int_{\mathfrak{a}^+} \chi_{e^{\mathfrak{a}(l)} o}(e^H o) \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^{m_\alpha} dH \\
&= const \cdot \int_{\mathfrak{a}(l)} \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^{m_\alpha} dH\,.
\end{aligned}$$

Furthermore, for $H \in \mathfrak{a}^+$ such that $\alpha(H) \geq const > 0$ for all $\alpha \in \Sigma^+$ we have

$$\prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^{m_\alpha} \asymp \prod_{\alpha \in \Sigma^+} e^{m_\alpha \alpha(H)} = e^{\sum_{\alpha \in \Sigma^+} m_\alpha \alpha(H)} = e^{\rho(H)}\,.$$

Since ρ is a linear functional on $\mathfrak{a}$, the Cauchy-Schwarz inegality implies $|\rho(H) - \langle H_\xi, H\rangle \rho(H_\xi)| \le ||\rho|| C$ for $H \in \mathfrak{a}(l)$, hence

$$|\rho(H) - ||H||\rho(H_\xi)| \le 2C||\rho||\,, \tag{8}$$

and we conclude vol $C(l) \asymp \int_{\mathfrak{a}(l)} e^{\rho(H_\xi)||H||} dH$.

Recall that r denotes the rank of X. Using spherical coordinates

$$\hat{H} = H/||H||\,, \quad t = ||H||$$

and the fact that

$$\int_{\substack{\hat{H}\in\mathfrak{a}_1^+ \\ \sin\angle(\hat{H},H_\xi)<\tau}} d\hat{H} = M\cdot\tau^{r-1}$$

with a constant M depending only on r, we conclude that for $l > C+s$

$$\begin{aligned} \text{vol } C(l) &\asymp \int_{\mathfrak{a}(l)} e^{\rho(H_\xi)||H||} dH = \int_{l-s}^{l} \Big(\int_{\substack{\hat{H}\in\mathfrak{a}_1^+ \\ \sin\angle(\hat{H},H_\xi)<C/t}} e^{\rho(H_\xi)t} d\hat{H}\Big) t^{r-1} dt \\ &= MC^{r-1}\int_{l-s}^{l} e^{\rho(H_\xi)t} dt \asymp e^{\rho(H_\xi)l}\,. \end{aligned}$$

For $\varepsilon > 0$ sufficiently small we put $\mathfrak{a}_\varepsilon := \{H \in \mathfrak{a} \mid ||H|| < \varepsilon\}$,

$$\begin{aligned} R_\varepsilon(l) &:= \{z\in X \mid l-\varepsilon \le d(o,z) \le l+\varepsilon,\ d(z,\sigma_{o,G\xi}) \le C+\varepsilon\}\,, \\ Q_\varepsilon(l) &:= \{z\in X \mid l-s-\varepsilon \le d(o,z) \le l+\varepsilon,\ C-\varepsilon \le d(z,\sigma_{o,G\xi}) \le C+\varepsilon\}\,. \end{aligned}$$

Then for $U_\varepsilon := Ke^{\mathfrak{a}_\varepsilon} \subset G$ we have $U_\varepsilon\cdot\partial C(l) \subseteq R_\varepsilon(l-s)\cup R_\varepsilon(l)\cup Q_\varepsilon(l)$. As above we compute

$$\begin{aligned} \text{vol } R_\varepsilon(l) &\asymp M(C+\varepsilon)^{r-1}\int_{l-\varepsilon}^{l+\varepsilon} e^{\rho(H_\xi)t}dt \\ &= \frac{M(C+\varepsilon)^{r-1}}{\rho(H_\xi)}\big(e^{\rho(H_\xi)\varepsilon} - e^{-\rho(H_\xi)\varepsilon}\big)e^{\rho(H_\xi)l} \\ &\asymp (C+\varepsilon)^{r-1}\big(e^{\rho(H_\xi)\varepsilon} - e^{-\rho(H_\xi)\varepsilon}\big)e^{\rho(H_\xi)l}\,, \\ \text{vol } Q_\varepsilon(l) &\asymp M\int_{l-s-\varepsilon}^{l+\varepsilon}\Big(\big(\frac{C+\varepsilon}{t}\big)^{r-1} - \big(\frac{C-\varepsilon}{t}\big)^{r-1}\Big)e^{\rho(H_\xi)t}t^{r-1}dt \\ &= \frac{M}{\rho(H_\xi)}\big((C+\varepsilon)^{r-1} - (C-\varepsilon)^{r-1}\big)\big(e^{\rho(H_\xi)\varepsilon} - e^{\rho(H_\xi)(-s-\varepsilon)}\big)e^{\rho(H_\xi)l} \\ &\asymp \big(e^{\rho(H_\xi)\varepsilon} - e^{-\rho(H_\xi)(s+\varepsilon)}\big)\big((C+\varepsilon)^{r-1} - (C-\varepsilon)^{r-1}\big)e^{\rho(H_\xi)l}\,. \end{aligned}$$

As ε tends to zero, the terms $(C+\varepsilon)^{r-1}\big(e^{\rho(H_\xi)\varepsilon} - e^{-\rho(H_\xi)\varepsilon}\big)$ and $\big(e^{\rho(H_\xi)\varepsilon} - e^{-\rho(H_\xi)(s+\varepsilon)}\big)\big((C+\varepsilon)^{r-1} - (C-\varepsilon)^{r-1}\big)$ both tend to zero. Hence there exists $\varepsilon > 0$ depending on φ, r, C, s, H_ξ such that for any $l \in \mathbb{N}$ sufficiently large

$$\frac{\text{vol } U_\varepsilon\partial C(l)}{\text{vol } C(l)} \le \frac{\text{vol } R_\varepsilon(l-s) + \text{vol } R_\varepsilon(l) + \text{vol } Q_\varepsilon(l)}{\text{vol } C(l)} < \varphi\,. \qquad \square$$

Proposition 4.2 then gives

$$N^C_{\Gamma\xi}(l) = \#\{\gamma \in \Gamma(G\cdot\xi, C) \mid l - s \le d(o,\gamma o) < l\} \asymp \mathrm{vol}\, C(l) \asymp e^{\rho(H_\xi)l}\,.$$

Theorem 4.4 *Let $\Gamma \subset G$ be an irreducible lattice, $\xi \in \partial X^{reg}$, and μ the $(b, \Gamma\cdot\xi)$-density constructed in [Li]. Then $\mu_o(L^{ray}_\Gamma \cap G\cdot\xi) = \mu_o(\partial X)$.*

Proof. By Theorem 3.10 it suffices to prove $\mu_o(L^{ray}_\Gamma \cap G\cdot\xi) = \mu_o(L^{ray}_\Gamma) > 0$. We claim that the criterium given in Proposition 4.1 is satisfied.

For l sufficiently large we are going to estimate

$$S(l) = \sum_{\substack{\gamma\in\Gamma(G\xi,C)\\ l-s\le d(o,\gamma o)<l}} e^{-\sum_{i=1}^r b^i d_i(o,\gamma o)}\,.$$

Denote by $H_\gamma \in \overline{\mathfrak{a}^+}$ the Cartan projection of γo. For $\gamma \in \Gamma(G\cdot\xi, C)$ we have

$$d_i(o,\gamma o) = \langle H_i, H_\gamma\rangle < d(o,\gamma o)\langle H_i, H_\xi\rangle + 2C\,.$$

Using the fact that the $(b, \Gamma\cdot\xi)$-density satisfies $\sum_{i=1}^r b^i\langle H_i, H_\xi\rangle = \rho(H_\xi)$ (see e.g. [Li, section 4.2]) we conclude that for l sufficiently large

$$\begin{aligned} S(l) &\ge \sum_{\substack{\gamma\in\Gamma(G\xi,C)\\ l-s\le d(o,\gamma o)<l}} e^{-\sum_{i=1}^r b^i\langle H_i,H_\xi\rangle d(o,\gamma o)-2C\|b\|_1} \\ &= e^{-2C\|b\|_1} \sum_{\substack{\gamma\in\Gamma(G\xi,C)\\ l-s\le d(o,\gamma o)<l}} e^{-\rho(H_\xi)d(o,\gamma o)} \\ &\ge e^{-2C\|b\|_1} N^C_{\Gamma\xi}(l) e^{-\rho(H_\xi)l} \ge const > 0\,, \end{aligned}$$

which finishes the proof. □

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Gabriele Link
Centre de Mathématiques Laurent Schwartz
Ecole Polytechnique
Route de Saclay
91 128 Palaisau
e-mail: gabi.link@gmx.de