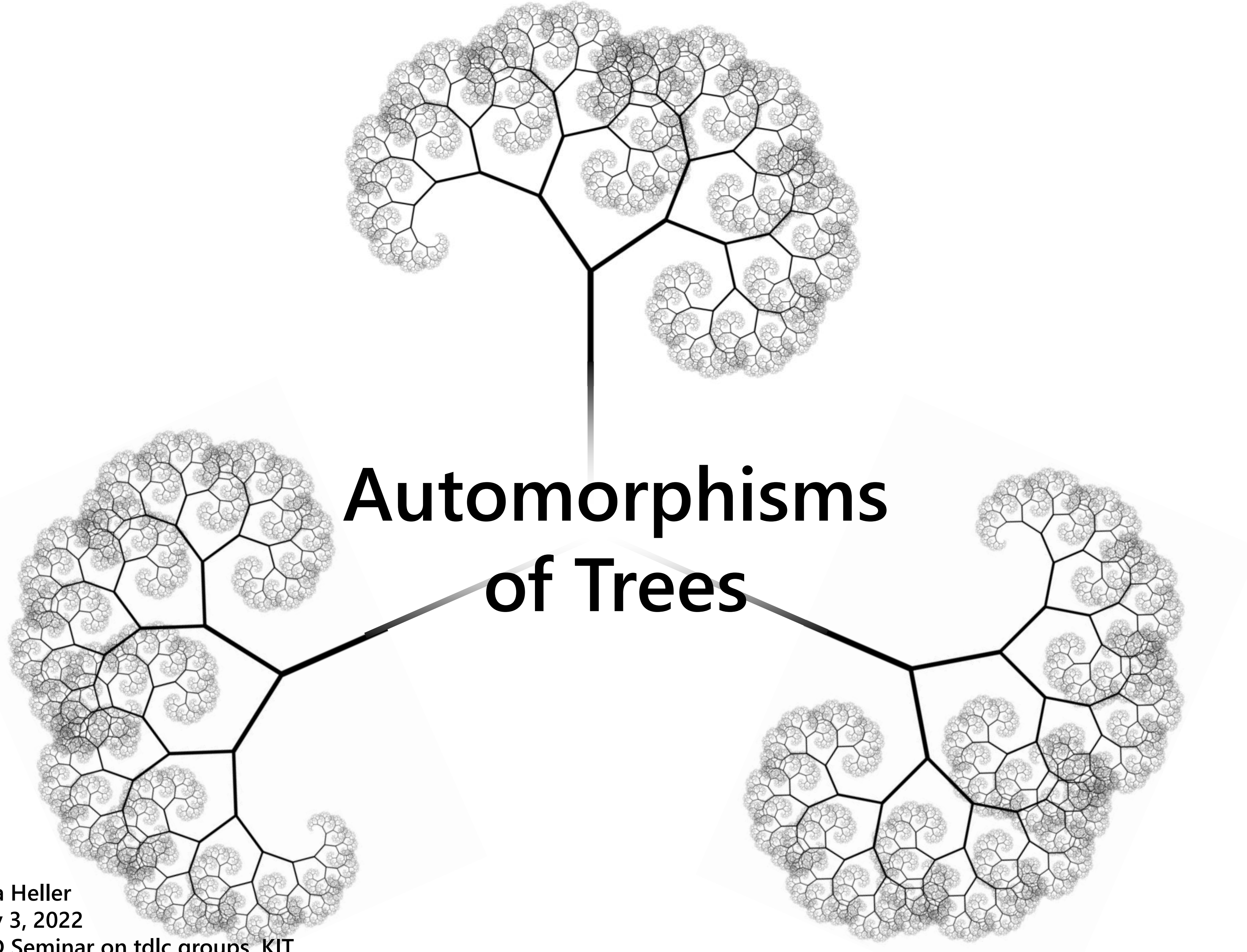


# Automorphisms of Trees

Julia Heller
May 3, 2022
GCD Seminar on tdlc groups, KIT

# What will happen?

trees

automorphisms
of trees

**universal
groups**

big picture
of tdlc groups

# … and why?

trees

automorphisms
of trees

**universal
groups**

big picture
of tdlc groups

## Overall goals

- find examples of tdlc groups
- understand tdlc groups
- classify tdlc groups

# … and why?

trees

automorphisms
of trees

universal
groups

big picture
of tdlc groups

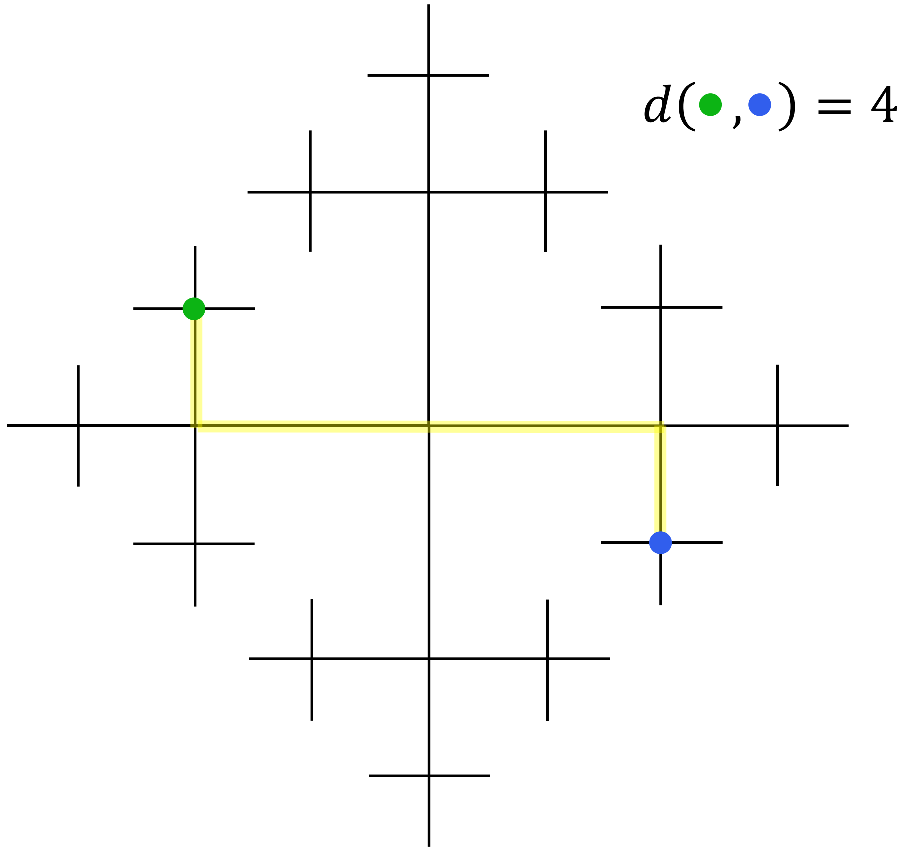
## Today's goals

Establish a first intuition for tdlc!

- proof that $\mathrm{Aut}(T)$ is tdlc
- draw a legal labeling
- decide if a given universal group is discrete or not

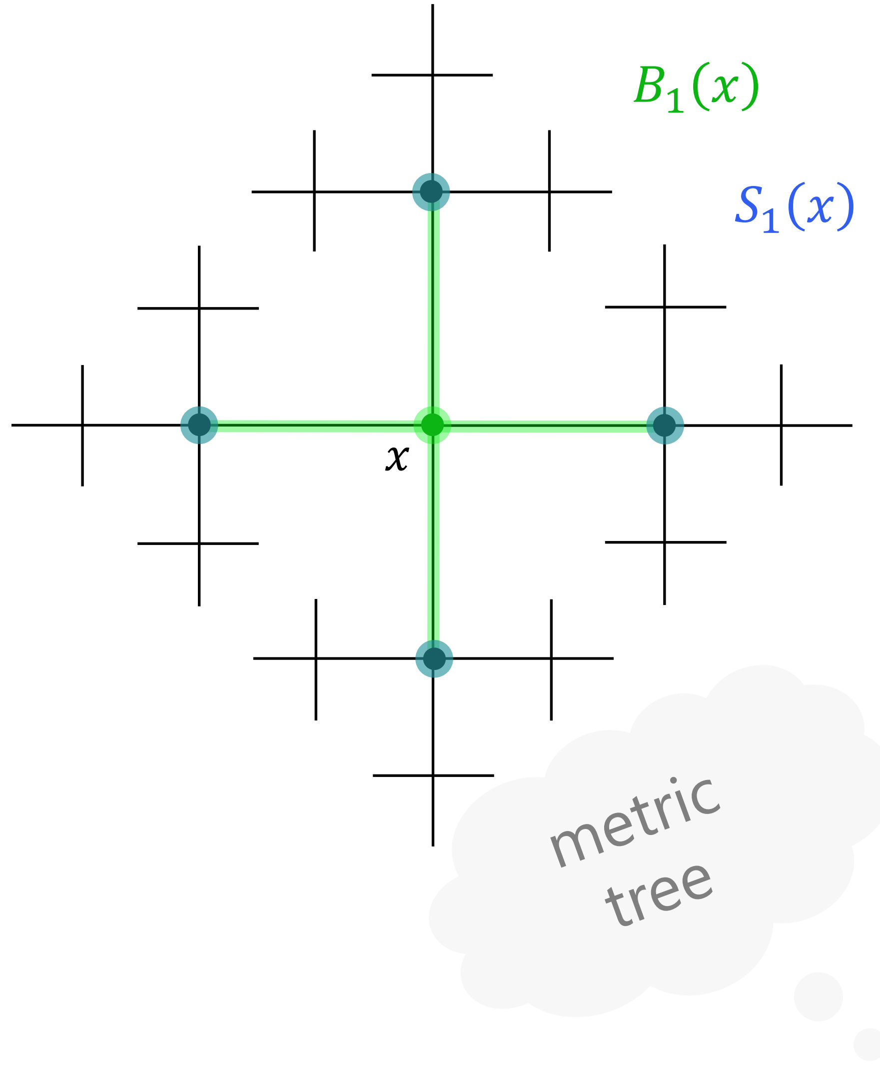
# Trees

- simplicial tree $T$
  vertices $V$
  edges $E \subset V \times V$
- metric on $T$
  edge length 1
- $T$ uniquely geodesic



# Trees

- simplicial tree $T$
  vertices $V$
  edges $E \subset V \times V$
- metric on $T$
  edge length 1
- $T$ uniquely geodesic
- ball $B_n(x)$
- sphere $S_n(x)$

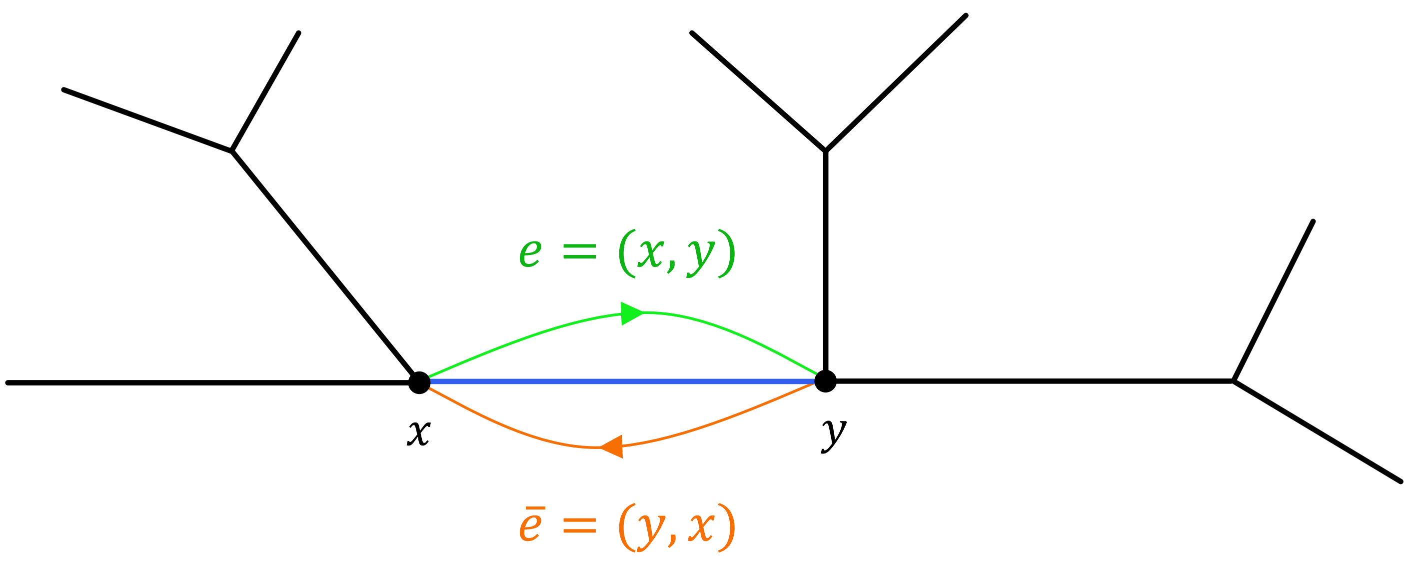


# Trees


- *oriented* edges $E \subset V \times V$

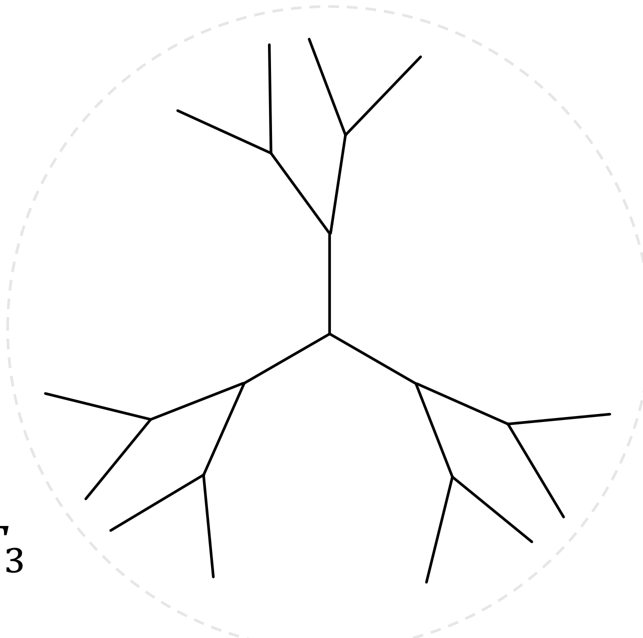
  one simplicial edge on $\{x, y\}$

  ⇝ two oriented edges $e$ and $\bar{e}$
- $o(e) = x$ *origin* of $e$, $t(e) = y$ *terminus* of $e$
- $E(x) \coloneqq \{e \in E : o(e) = x\}$

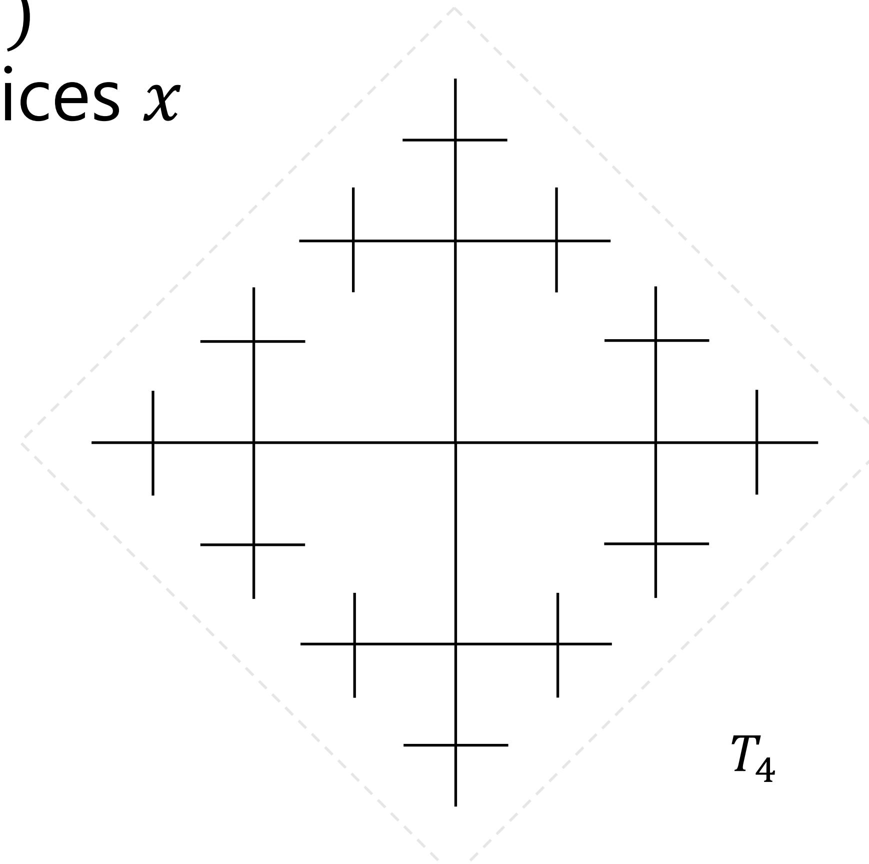


# Trees of interest

$T_d$: $d$-regular tree ($d \geq 3$)
$\#E(x) = d$ for all vertices $x$



$T_3$

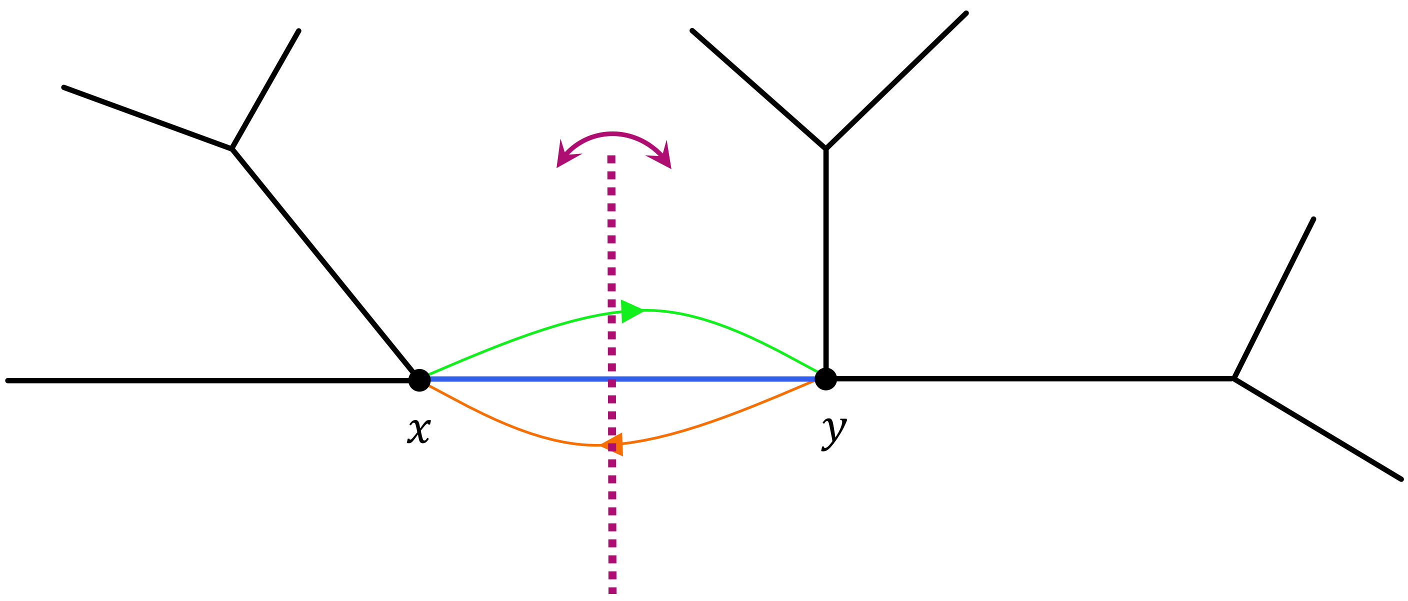


$T_4$

**From now on:** $T = T_d$ regular tree with $d \geq 3$

# Trees – remarks

- metric trees ⟺ combinatorial trees
- why oriented edges?
  - edge inversions should *flip*, not *fix*, edges



# Automorphisms of $T$

translations
inversions
rotations

$\mathrm{Aut}(T)$ *automorphism group* of $T$

$g \in \mathrm{Aut}(T)$ automorphism of $T$:

$g\colon V \to V$ bijection such that

$(x, y) \in E$ iff $(gx, gy) \in E$

# Automorphisms of $T$

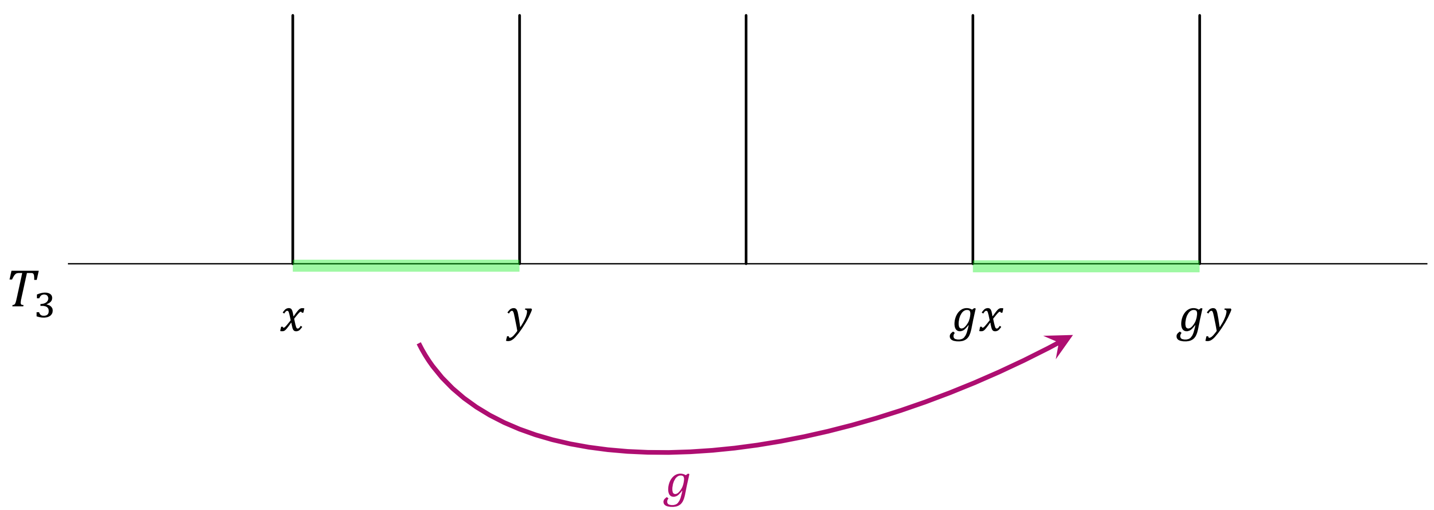
translations
inversions
rotations

$\mathrm{Aut}(T)$ *automorphism group* of $T$

$g \in \mathrm{Aut}(T)$ automorphism of $T$:

$g: V \to V$ bijection such that

$(x, y) \in E \quad \text{iff} \quad (gx, gy) \in E$



# Automorphisms of $T$

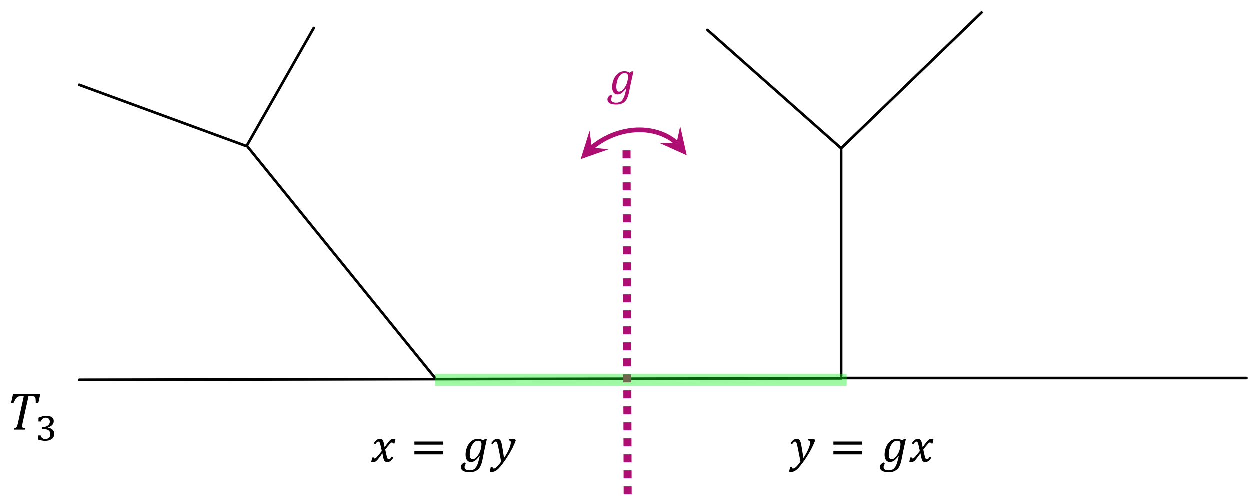
translations
inversions
rotations

$\mathrm{Aut}(T)$ *automorphism group* of $T$

$g \in \mathrm{Aut}(T)$ automorphism of $T$:

$g: V \to V$ bijection such that

$(x, y) \in E$ iff $(gx, gy) \in E$



# Automorphisms of $T$

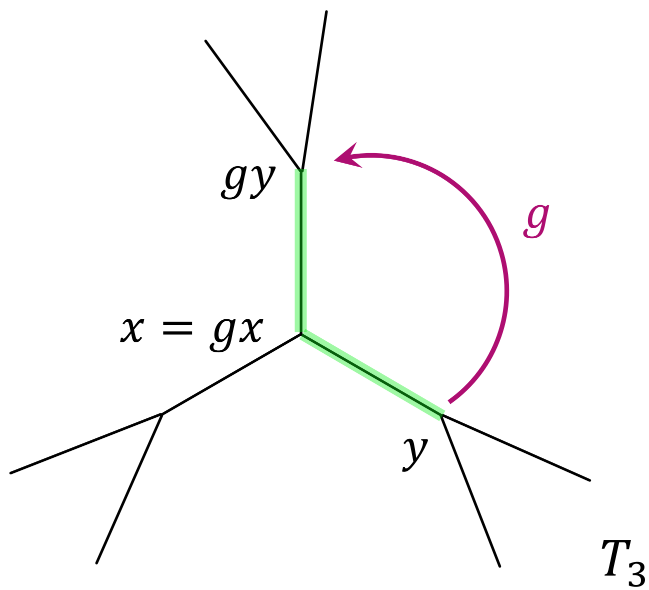
translations
inversions
rotations

$\mathrm{Aut}(T)$ *automorphism group* of $T$

$g \in \mathrm{Aut}(T)$ automorphism of $T$:

$g: V \to V$ bijection such that

$(x, y) \in E \quad \text{iff} \quad (gx, gy) \in E$



# Goal: $\mathrm{Aut}(T)$ is a tdlc group

- Plan
  - define topology on $\mathrm{Aut}(T)$
  - show that with this topology, $\mathrm{Aut}(T)$ is
    - totally disconnected
    - locally compact

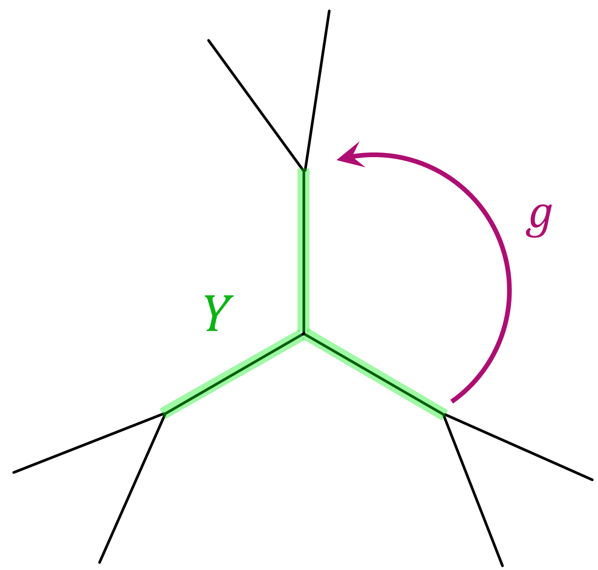
- Remarks
  - we use the *topology of pointwise convergence*
  - here, it coincides with the *compact-open topology*

# Towards the topology on $\mathrm{Aut}(T)$

$G \leq \mathrm{Aut}(T)$, $Y \subseteq T$

- *stabilizer* of $Y$: $\mathrm{St}_G(Y) \coloneqq \{g \in \mathrm{G} : \ gY = Y\}$
    - $g \in \mathrm{St}_G(Y)$

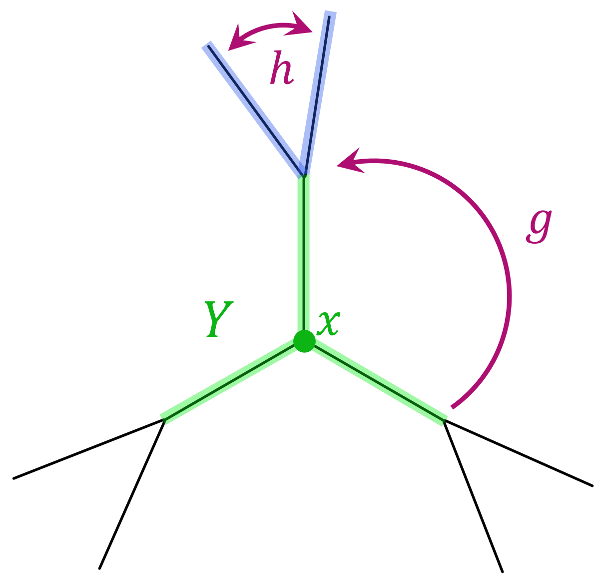
stabilizer – setwise



# Towards the topology on $\mathrm{Aut}(T)$

$G \leq \mathrm{Aut}(T)$, $Y \subseteq T$

- *stabilizer* of $Y$: $\mathrm{St}_G(Y) \coloneqq \{g \in G : \; gY = Y\}$
  - $g \in \mathrm{St}_G(Y)$
- *fixator* of $Y$: $\mathrm{Fix}_G(Y) \coloneqq \{g \in G : \; gy = y \;\; \forall y \in Y\}$
  - $g \notin \mathrm{Fix}_G(Y)$
  - $g \in \mathrm{Fix}_G(x) = \mathrm{St}_G(x)$
  - $h \in \mathrm{Fix}_G(Y) \subset \mathrm{Fix}_G(x)$




stabilizer – setwise
fixator – pointwise

# Basic open sets

- $g \in \mathrm{Aut}(T)$ automorphism
- $S \subset V$ *finite* set of vertices

$$U(g,S) \coloneqq \{h \in \mathrm{Aut}(T)\colon\ hx = gx \text{ for all } x \in S\}$$

$U(g,S)$ are *basic open sets* of the topology of pointwise convergence on vertices

Note: $U(\mathrm{id}, S) = \mathrm{Fix}(S)$

$g, h$ are close if they agree on a large subset

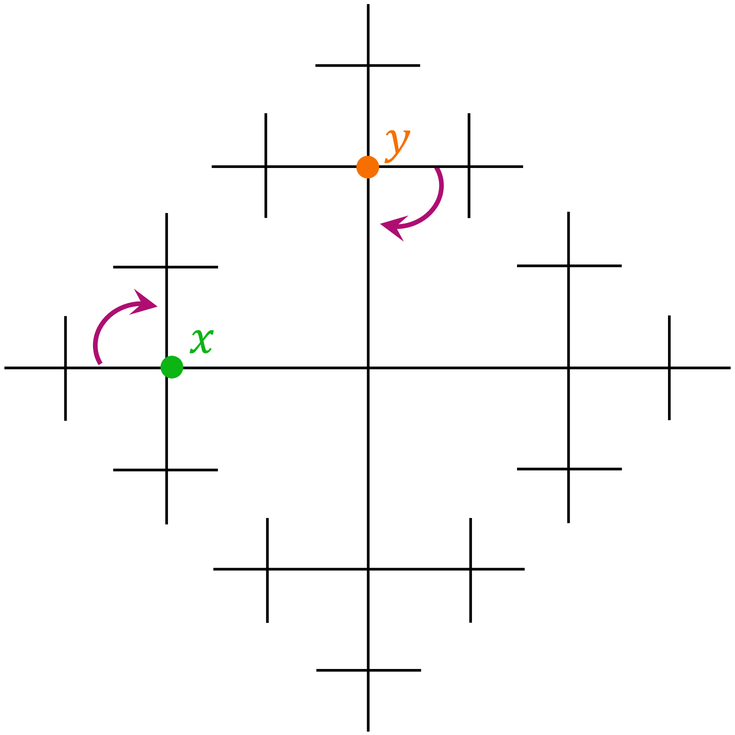
$\mathrm{Fix}_G(Y) = \{g \in G:\ gy = y\ \ \forall y \in Y\}$

# Open sets ($G = \mathrm{Aut}(T)$)

$\mathrm{Fix}(S) = U(\mathrm{id}, S) = \{h \in G\colon\ hx = x \ \ \forall x \in S\}$

- $U(\mathrm{id}, x) = \{h \in G\colon\ hx = x\}$
- $U(\mathrm{id}, y) = \{h \in G\colon\ hy = y\}$

*unions* of basic sets
$U(\mathrm{id}, x) \cup U(\mathrm{id}, y)$
$= \{h \in G\colon\ hx = x \text{ or } hy = y\}$
*not* of the form $U(\mathrm{id}, S)$




split open sets along their fixed sets

# Open sets ($G = \mathrm{Aut}(T)$)

$\mathrm{Fix}(S) = U(\mathrm{id}, S) = \{h \in G\colon\ hx = x \ \ \forall x \in S\}$

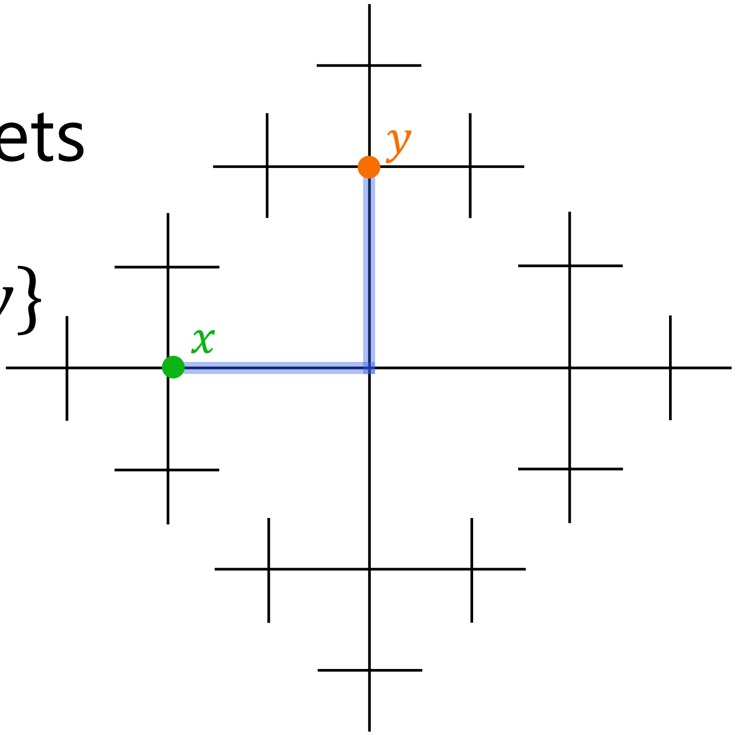
- $U(\mathrm{id}, x) = \{h \in G\colon\ hx = x\}$
- $U(\mathrm{id}, y) = \{h \in G\colon\ hy = y\}$

*finite intersections* of basic sets

$U(\mathrm{id}, x) \cap U(\mathrm{id}, y)$

$= \{h \in G\colon\ hx = x \text{ and } hy = y\}$

$= U(\mathrm{id}, \mathrm{conv}(x, y))$




basic sets with S
fixed subtree

# Open sets ($G = \mathrm{Aut}(T)$)

$\mathrm{Fix}(S) = U(\mathrm{id}, S) = \{h \in G\colon\ hx = x\ \ \forall x \in S\}$

- $U(\mathrm{id}, x) = \{h \in G\colon\ hx = x\}$
- $U(\mathrm{id}, y) = \{h \in G\colon\ hy = y\}$

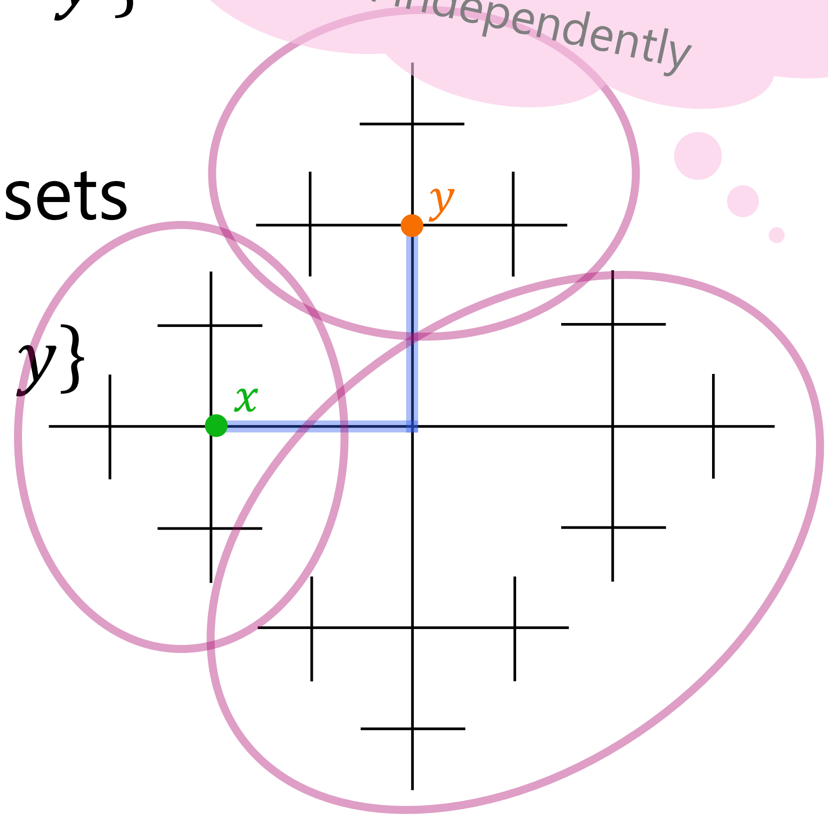
automorphisms outside S can be chosen independently

*finite intersections* of basic sets

$U(\mathrm{id}, x) \cap U(\mathrm{id}, y)$

$= \{h \in G\colon\ hx = x \text{ and } hy = y\}$

$= U(\mathrm{id}, \mathrm{conv}(x, y))$




basic sets with S fixed subtree

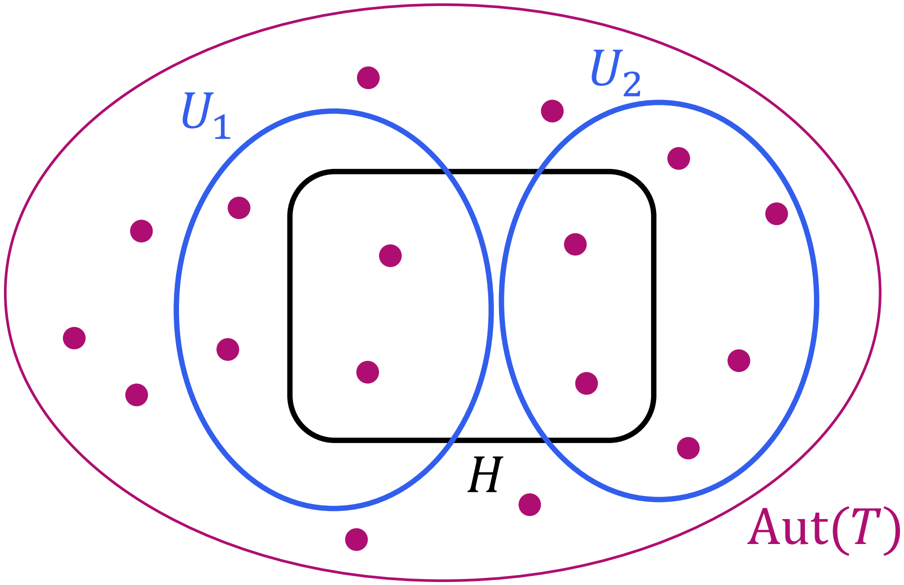
# $G = \mathrm{Aut}(T)$ is totally disconnected

$G$ is *totally disconnected* if
every connected set is a singleton

we show:
if $H \subseteq G$ with $\#H \geq 2$, then $H$ is disconnected, i.e.

there are open sets $U_1, U_2 \neq \emptyset$ in $\mathrm{Aut}(T)$ such that

- $U_1 \cap U_2 = \emptyset$
- $U_1 \cap H,\ U_2 \cap H \neq \emptyset$
- $(U_1 \cup U_2) \cap H = H$

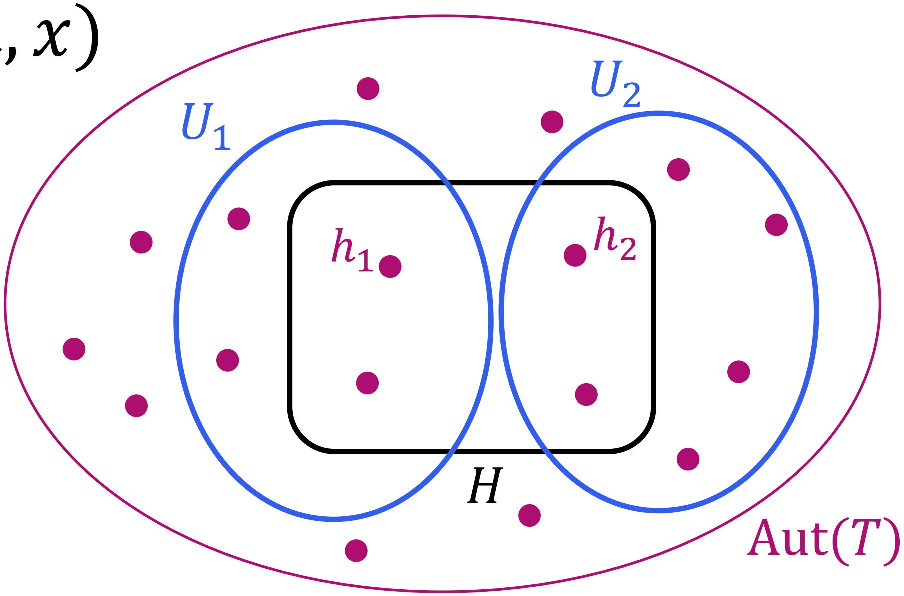


# $G = \mathrm{Aut}(T)$ is totally disconnected

- Let $H \subseteq G$ with $\# H \geq 2$
- choose $h_1, h_2 \in H$ and $x \in T$ such that $h_1 x \neq h_2 x$
- set $U_1 := U(h_1, x)$
- set $U_2 := \bigcup_{h \in H \setminus U_1} U(h, x)$

Then

- $U_1, U_2$ open in $G$
- $h_1 \in U_1$ , $h_2 \in U_2$
- $U_1 \cap U_2 = \emptyset$
- $(U_1 \cup U_2) \cap H = H$




Hence, $\mathrm{Aut}(T)$ is totally disconnected.

Moreover, $\mathrm{Aut}(T)$ is Hausdorff.

# $G = \mathrm{Aut}(T)$ is locally compact

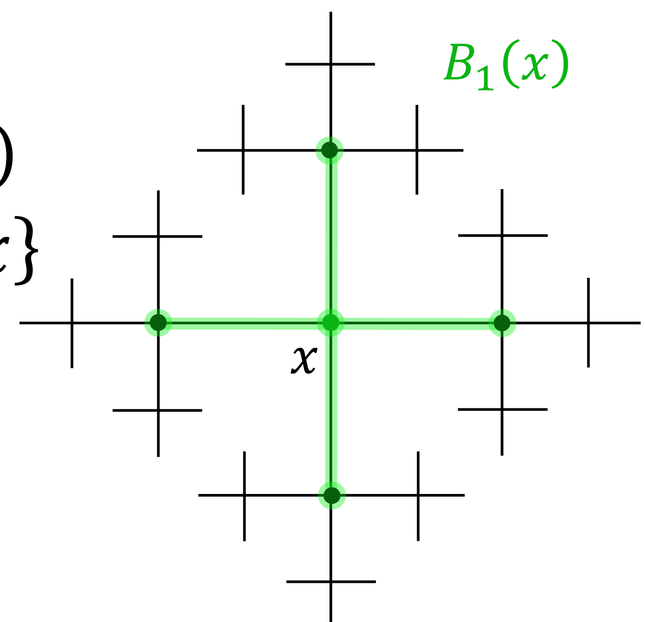
build autom. by extending maps from ball to ball successively

$G$ is *locally compact* if
$\mathrm{id} \in G$ has a compact neighborhood

<u>we show:</u>

$\mathrm{St}(x) = U(\mathrm{id}, x)$ is a compact neighborhood of id

- $g \in \mathrm{St}(x)$: $gB_n(x) = B_n(x)$
- restrict $\mathrm{St}(x)$ to finite balls $B_n(x)$
  - $Y_n := \{h \in \mathrm{Aut}(B_n(x)) : hx = x\}$
  - $Y_n$ is finite



# $G = \mathrm{Aut}(T)$ is locally compact

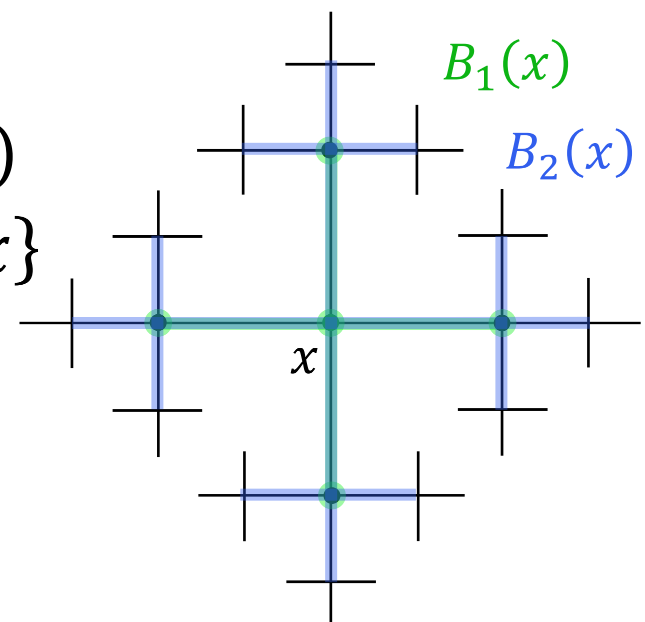
build autom. by extending maps from ball to ball successively

$G$ is *locally compact* if
$\mathrm{id} \in G$ has a compact neighborhood

<u>we show:</u>

$\mathrm{St}(x) = U(\mathrm{id}, x)$ is a compact neighborhood of id

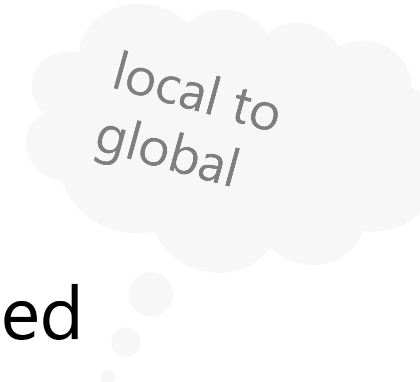
- $g \in \mathrm{St}(x)$: $gB_n(x) = B_n(x)$
- restrict $\mathrm{St}(x)$ to finite balls $B_n(x)$
  - $Y_n := \{h \in \mathrm{Aut}(B_n(x)) : hx = x\}$
  - $Y_n$ is finite
- $\mathrm{St}(x) = \varprojlim Y_n$ inverse limit of finite sets, hence compact



# So far, so good: Tree automorphisms

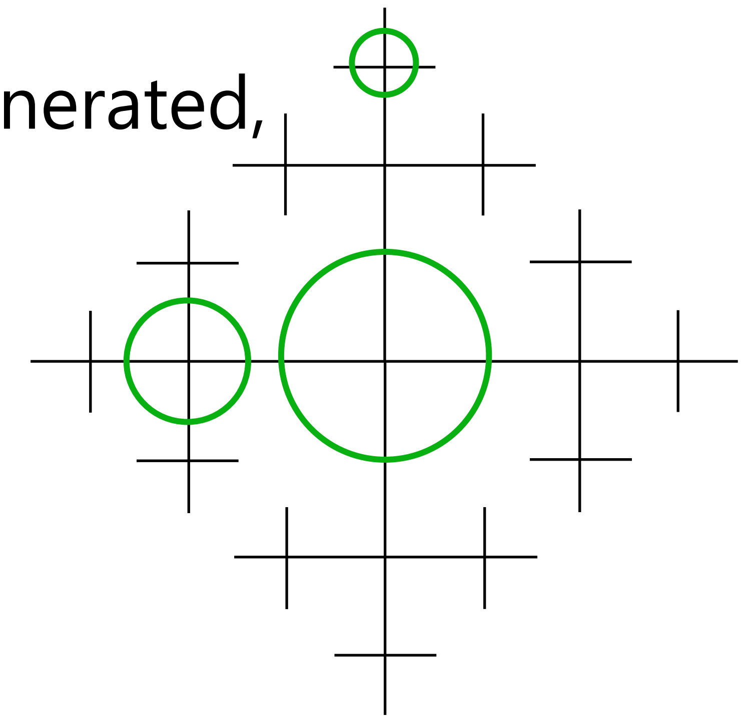
- metric and combinatorial trees
  - regular trees
- automorphisms of trees
- $\mathrm{Aut}(T)$ is a tdlc group

# Now: Universal groups




Burger-Mozes 2000:

- subgroups of $\mathrm{Aut}(T_d)$ with prescribed local action around vertices
- local-to-global properties
  - universality
- source of compactly generated, simple groups
  - building blocks for a possible classification



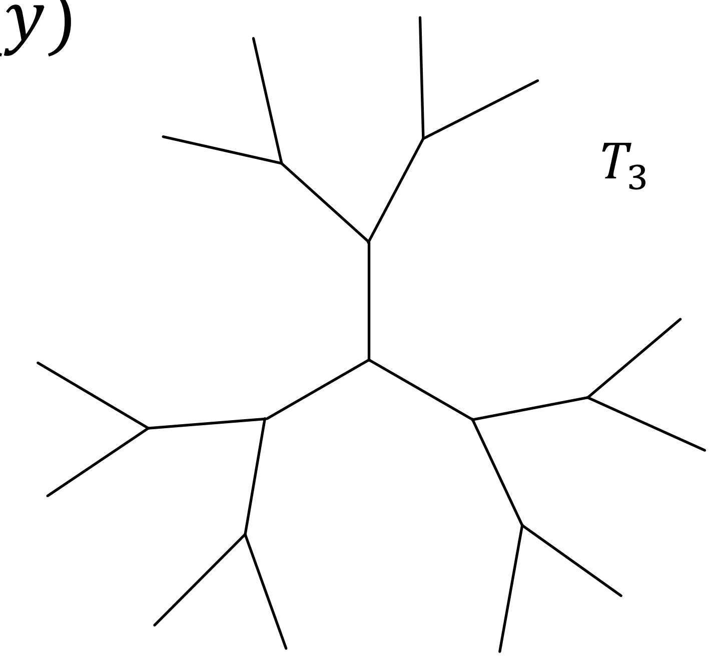
# Legal labelings

$T = T_d$ regular tree

$l: E \to \{1, \ldots, d\}$ is a *legal labeling* of $T$ if

- $l(y) = l(\bar{y})$ for all $y \in E$
- $l_x: E(x) \to \{1, \ldots, d\}, y \mapsto l(y)$ is a bijection for all $x \in V$

$E(x)$: edges starting at x



# Legal labelings

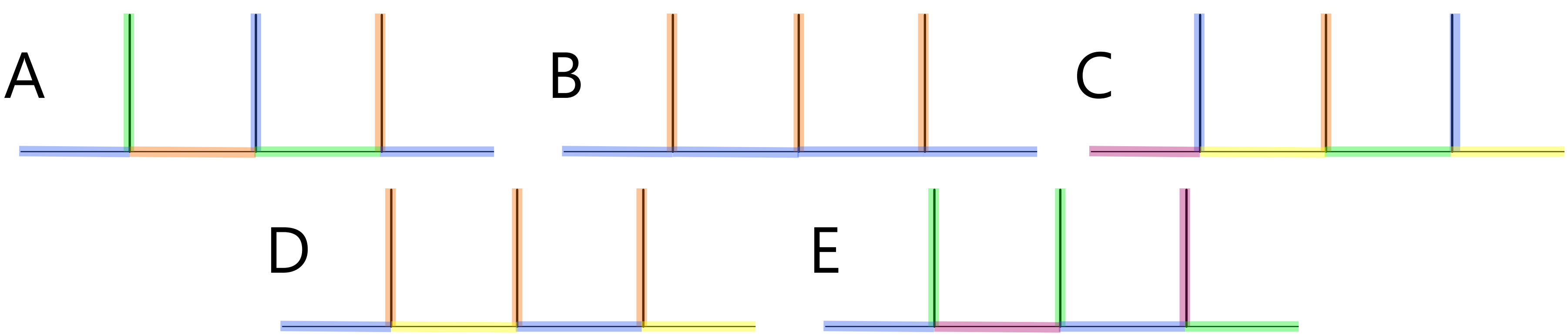
$E(x)$: edges starting at x

$T = T_d$ regular tree

$l\colon E \to \{1, \dots, d\}$ is a *legal labeling* of $T$ if

- $l(y) = l(\bar{y})$ for all $y \in E$
- $l_x\colon E(x) \to \{1, \dots, d\}, y \mapsto l(y)$ is a bijection for all $x \in V$

Which (partial) labeling of $T_3$ is legal?



# Legal labelings

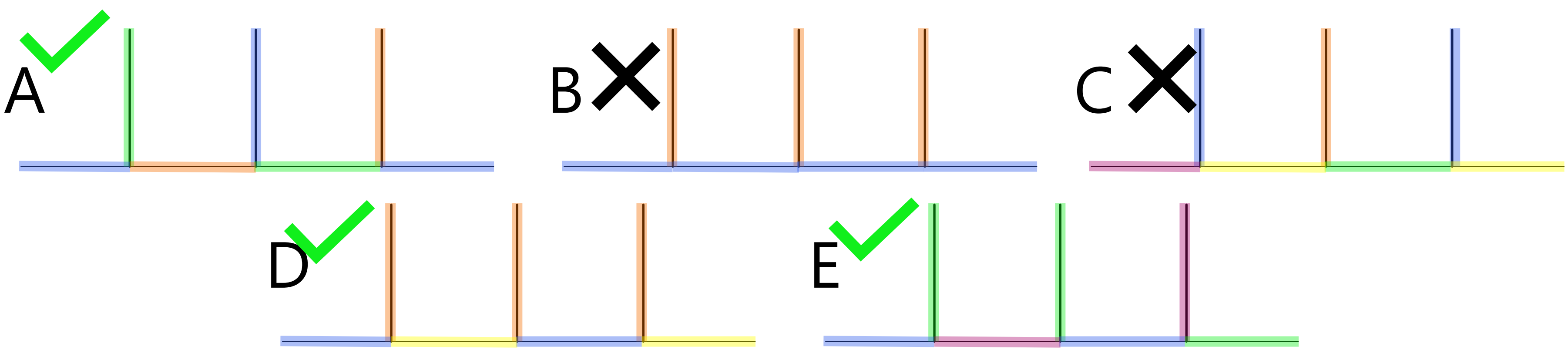
$E(x)$: edges starting at x

$T = T_d$ regular tree

$l: E \to \{1, \dots, d\}$ is a *legal labeling* of $T$ if

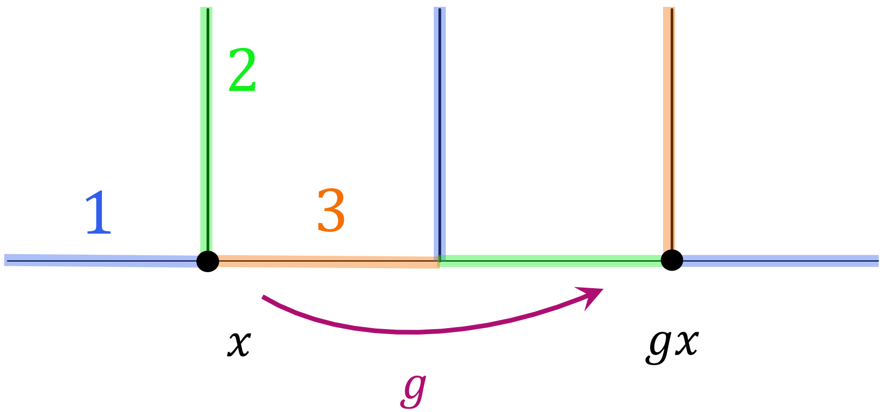
- $l(y) = l(\bar{y})$ for all $y \in E$
- $l_x: E(x) \to \{1, \dots, d\}, y \mapsto l(y)$
  is a bijection for all $x \in V$

Which (partial) labeling of $T_3$ is legal?



# Induced local permutation

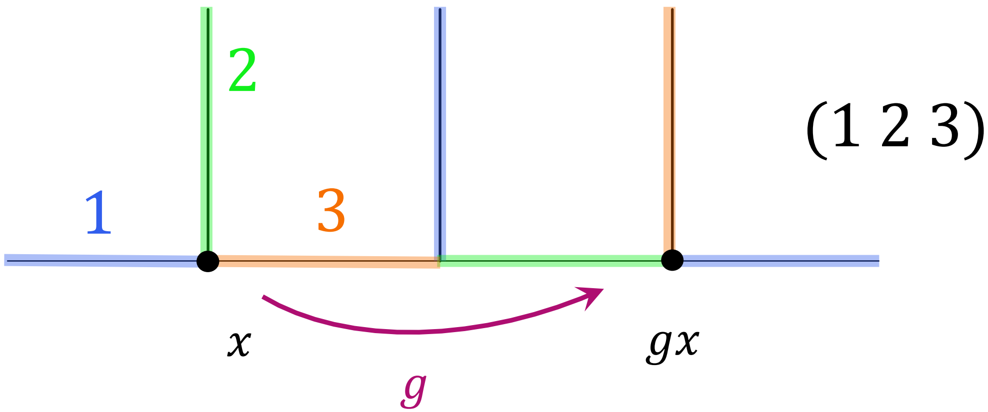
input: $l$ legal labeling, $g \in \mathrm{Aut}(T)$, $x \in V$




output: *induced local permutation* $c(g, x)$ in $S_d$

# Induced local permutation

input: $l$ legal labeling, $g \in \mathrm{Aut}(T)$, $x \in V$



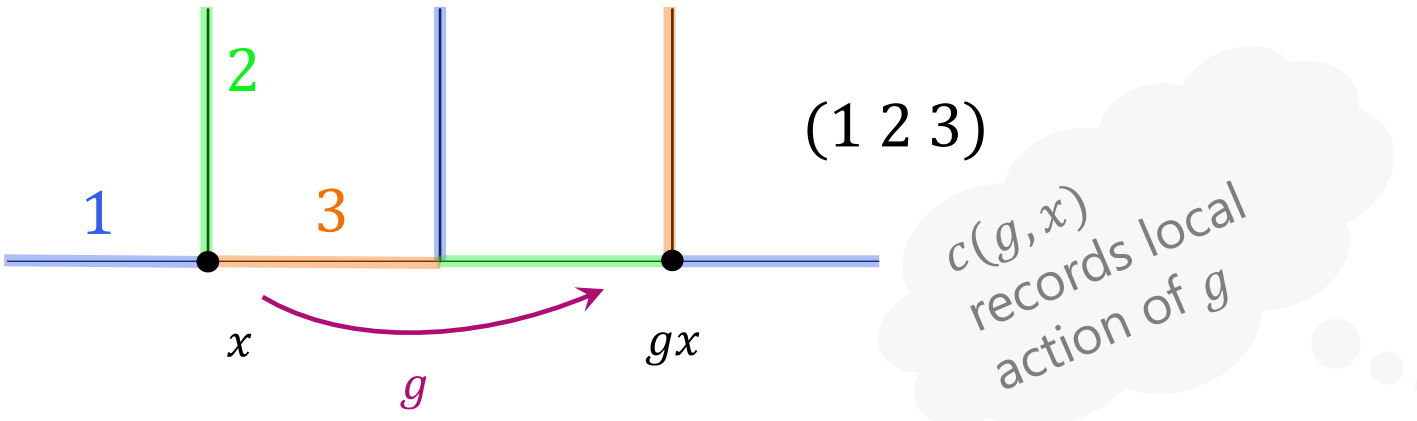

output: *induced local permutation* $c(g,x)$ in $S_d$

$$c\colon \mathrm{Aut}(T) \times V \to S_d$$
$$(g,x) \mapsto l_{gx} \circ g \circ l_x^{-1} = c(g,x)$$

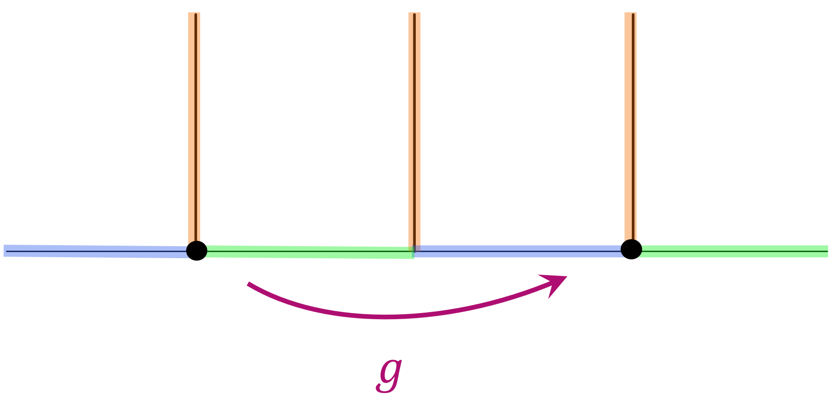
$$c(g,x)\colon \{1,\ldots,d\} \to E(x) \to E(gx) \to \{1,\ldots,d\}$$

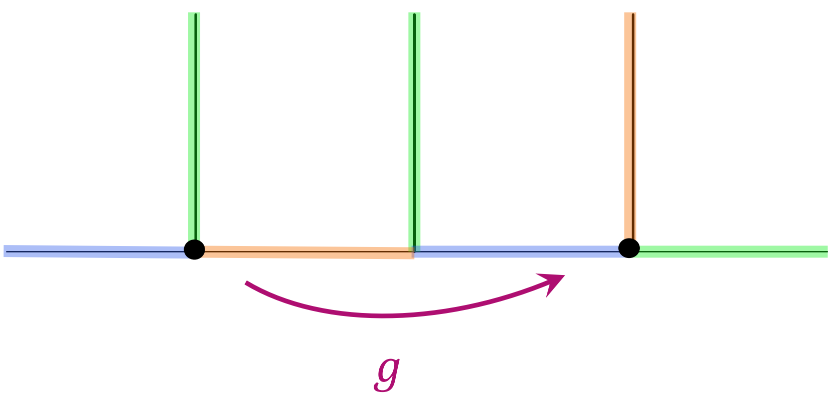
# Induced local permutation

input: $l$ legal labeling, $g \in \mathrm{Aut}(T)$, $x \in V$



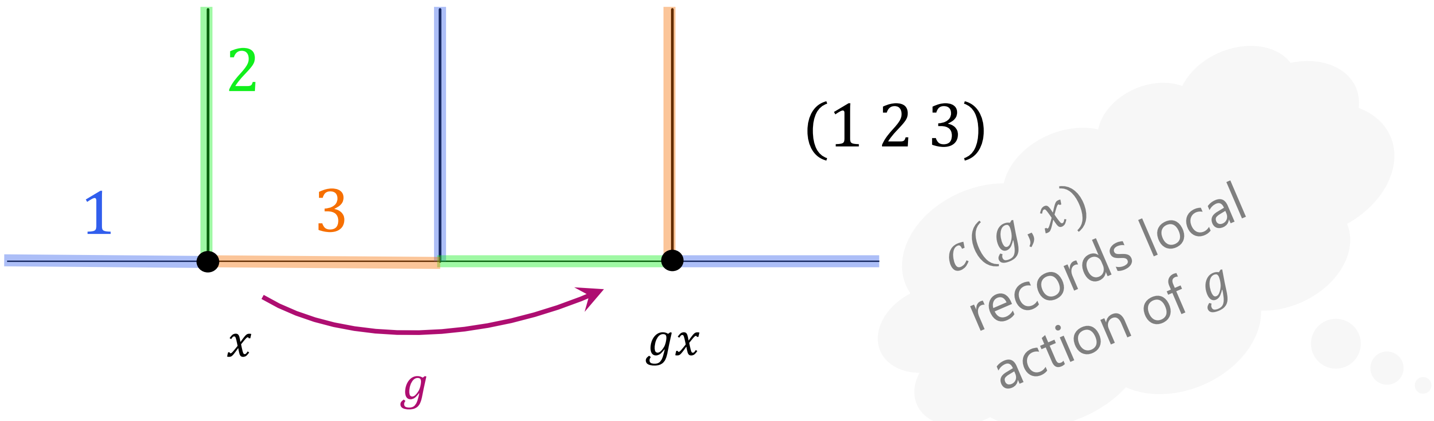

What is $c(g,x) = l_{gx} \circ g \circ l_x^{-1}$ here?



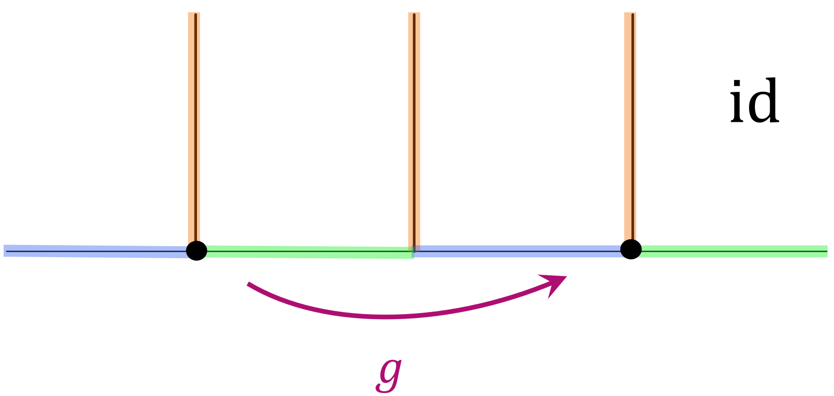



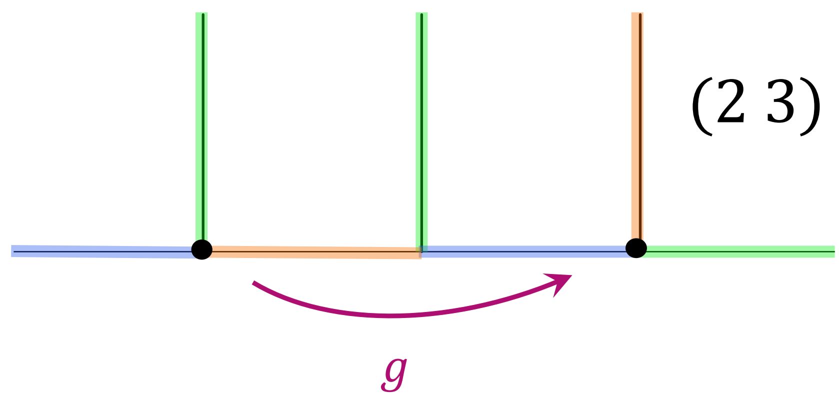
# Induced local permutation

input: $l$ legal labeling, $g \in \mathrm{Aut}(T)$, $x \in V$



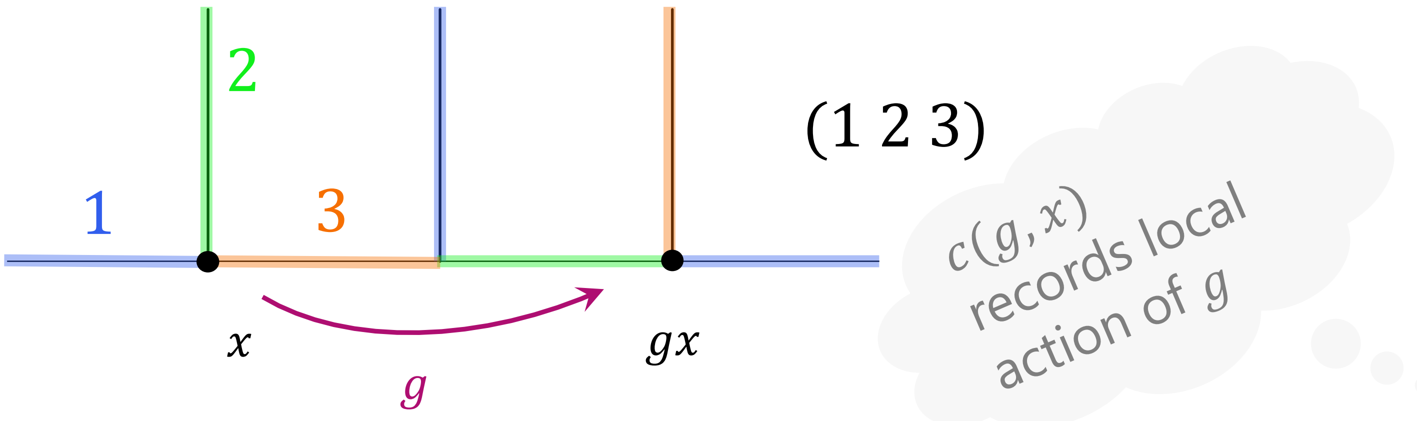

What is $c(g,x) = l_{gx} \circ g \circ l_x^{-1}$ here?



# Induced local permutation

input: $l$ legal labeling, $g \in \mathrm{Aut}(T)$, $x \in V$




set of *induced local permutations* for $g$:
$$\{c(g,x): x \in V\} \subseteq S_d$$

# Universal groups

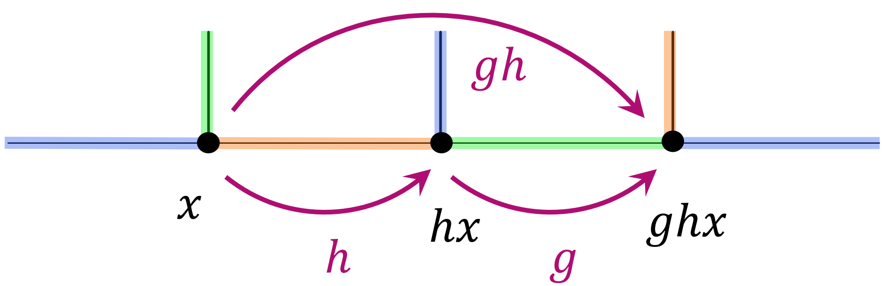
Let $F \leq S_d$ and $l$ legal labeling, define

$$U^l(F) \coloneqq \{g \in \mathrm{Aut}(T) \colon c(g,x) \in F \text{ for all } x \in V\}$$

<u>Lemma</u>: $U^l(F)$ is a group.

<u>Proof</u>: to show: If $g, h \in U^l(F)$, then $gh \in U^l(F)$, i.e. $c(gh, x) \in F$.

We have $c(gh,x) = c(g,hx)c(h,x) \in F$.



# Universal groups

- $U^l(F)$ is called *universal group*
- $F$ is called the *local action*

Remarks

- $g \in U^l(F)$ if all its local permutations are in $F$:
$$\{c(g,x) : x \in V\} \subseteq F \leq S_d$$
- $U^l(F') \leq U^l(F)$ for $F' \leq F$

# Independence of labelings

Lemma

Let $(l, l', x, x')$ with $l, l'$ legal labelings and $x, x' \in V$, then there is a unique $g \in \mathrm{Aut}(T)$ such that

$$gx = x' \text{ and } l' = l \circ g.$$

Proof

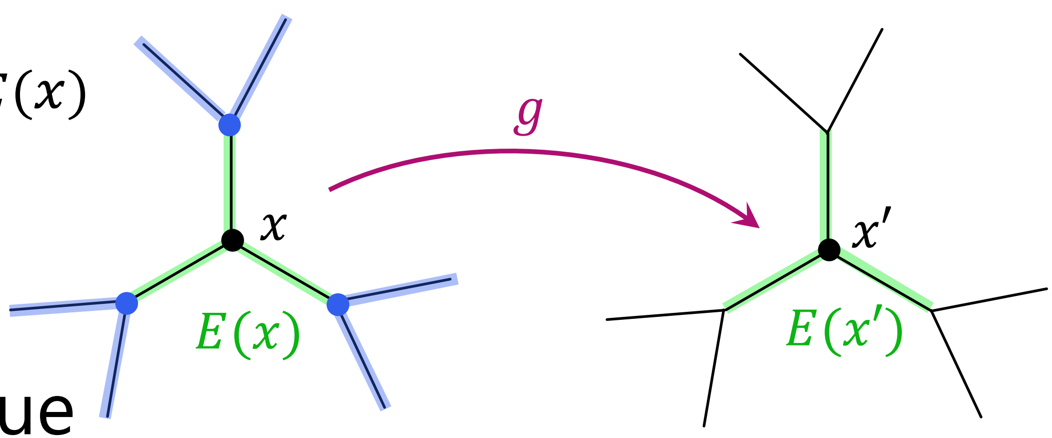
$gx = x'$ by assumption

$l'|_{E(x)} = l|_{E(x')} \circ g|_{E(x)}$

bijections

$\rightsquigarrow g|_{E(x)}$ unique

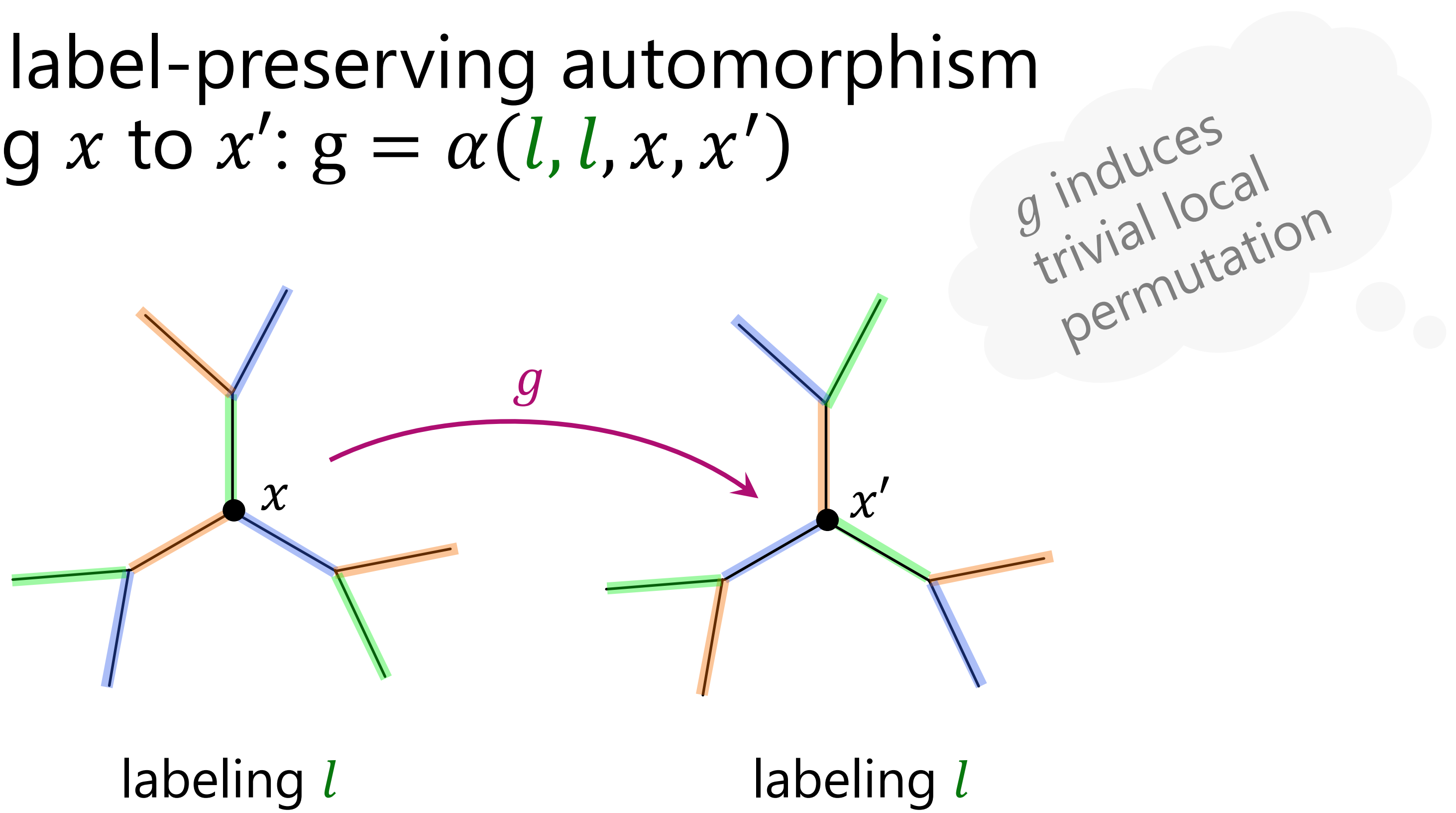
induction $\rightsquigarrow g$ unique



# Independence of labelings

$$\alpha\colon (l, l', x, x') \mapsto g = \alpha(l, l', x, x') \in \mathrm{Aut}(T)$$

Unique label-preserving automorphism mapping $x$ to $x'$: g $= \alpha(l, l, x, x')$

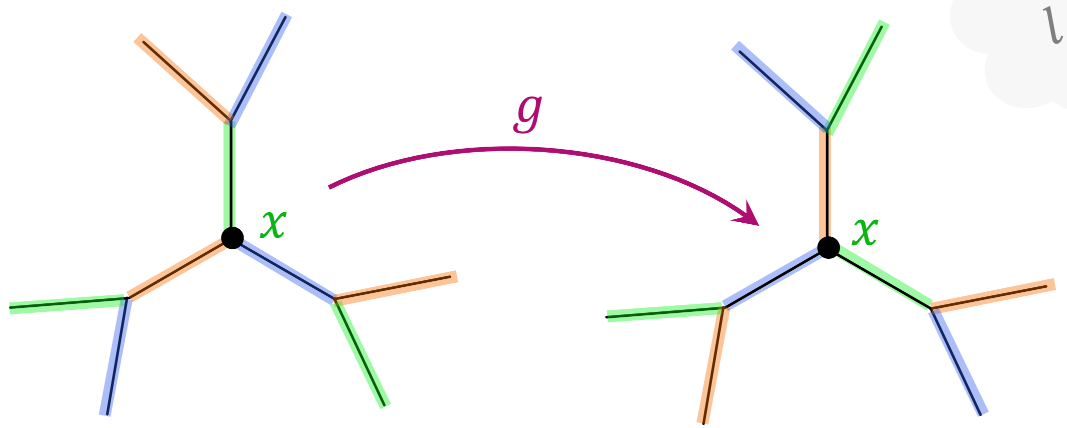


# Independence of labelings

$$\alpha\colon (l, l', x, x') \mapsto g = \alpha(l, l', x, x') \in \mathrm{Aut}(T)$$

Unique label-changing automorphism
fixing $x \in V$: $g = \alpha(l, l', x, x)$

$g$ transfers $l$ to $l'$ at $x$




labeling $l$ labeling $l'$

# Independence of labelings

The universal groups $U^l(F)$ and $U^{l'}(F)$ are conjugate in $\mathrm{Aut}(T)$:

Let $g = \alpha(l, l', x, x)$ for a $x \in V$, then

$$U^l(F) = gU^{l'}(F)g^{-1}$$

Hence: $U(F)$ is "the" universal group for $F \leq S_d$ with fixed (arbitrary) legal labeling $l$

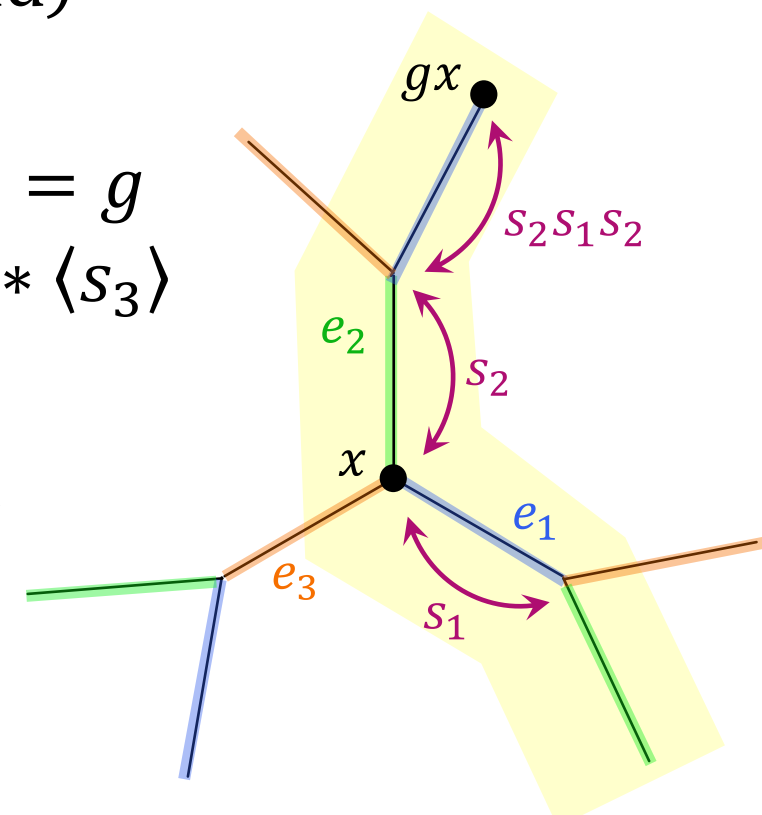
Examples

- $U(S_d) = \mathrm{Aut}(T)$
- $U(\mathrm{id}) \cong \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$, $d$ factors

# $U(\mathrm{id}) \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$

infinite dihedral group

- $x \in V$, $E(x) = \{e_1, e_2, e_3\}$
- $s_i = \alpha(l, l, o(e_i), t(e_i))$ edge inversions
- $\langle s_1 \rangle * \langle s_2 \rangle * \langle s_3 \rangle \leq U(\mathrm{id})$
- $g \in U(\mathrm{id})$
  - $\alpha(l, l, x, gx) = s_2 s_1 = g$
  - $U(\mathrm{id}) \leq \langle s_1 \rangle * \langle s_2 \rangle * \langle s_3 \rangle$




Automorphisms of this kind are contained in all universal groups as

$U(\mathrm{id}) \leq U(F)$

# Properties of universal groups

A universal group $U(F)$ is

- vertex-transitive
- transitive on edges of the same color
- closed in $\mathrm{Aut}(T)$
  - as such a tdlc Hausdorff group
- compactly generated
- discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1, \dots, d\}$ is free

local to global

# Properties of universal groups

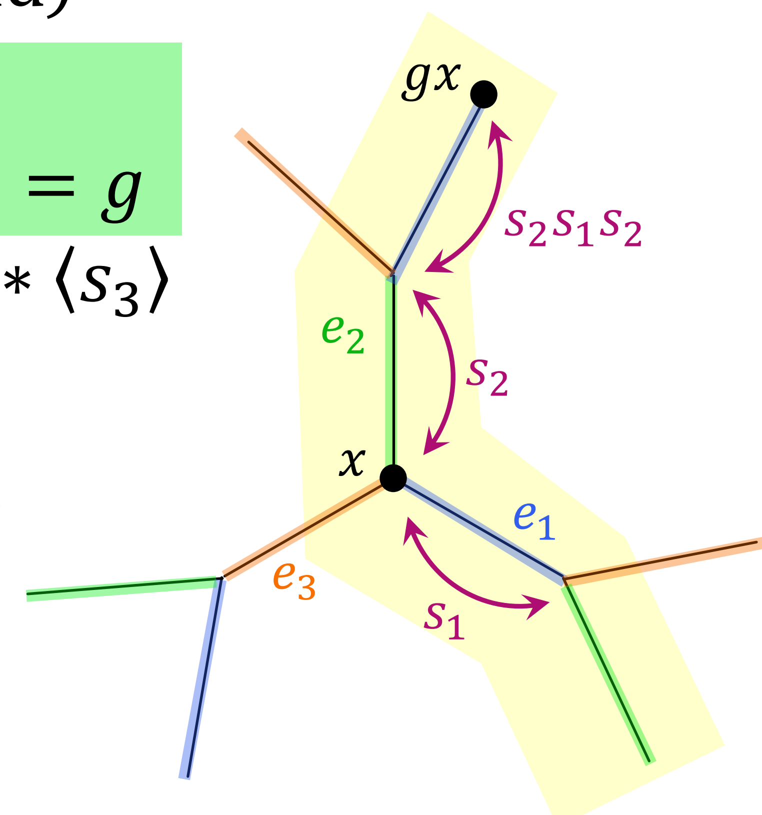

A universal group $U(F)$ is

- **vertex-transitive**
- transitive on edges of the same color
- closed in $\mathrm{Aut}(T)$
  - as such a tdlc Hausdorff group
- compactly generated
- discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1, \ldots, d\}$ is free

# $U(\text{id}) \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$

infinite dihedral group

- $x \in V$, $E(x) = \{e_1, e_2, e_3\}$
- $s_i = \alpha(l, l, o(e_i), t(e_i))$ edge inversions
- $\langle s_1 \rangle * \langle s_2 \rangle * \langle s_3 \rangle \leq U(\text{id})$
- $g \in U(\text{id})$
  - $\alpha(l, l, x, gx) = s_2 s_1 = g$
  - $U(\text{id}) \leq \langle s_1 \rangle * \langle s_2 \rangle * \langle s_3 \rangle$




Automorphisms of this kind are contained in all universal groups as

$U(\text{id}) \leq U(F)$

# Properties of universal groups

A universal group $U(F)$ is

- vertex-transitive
- **transitive on edges of the same color**
- closed in $\mathrm{Aut}(T)$
  - as such a tdlc Hausdorff group
- compactly generated
- discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1, \ldots, d\}$ is free

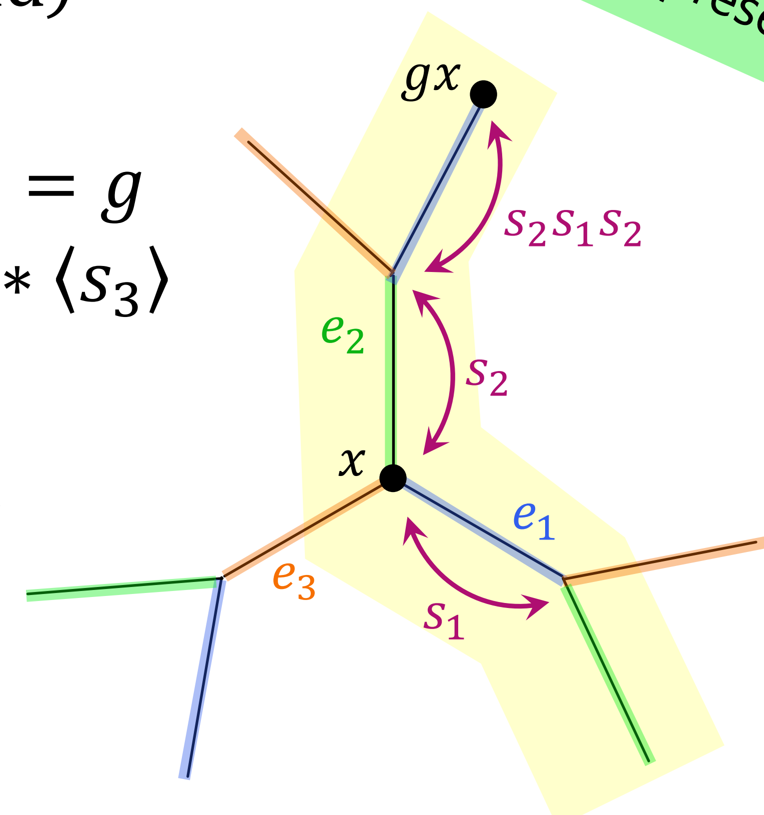
local to global

# $U(\text{id}) \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$

infinite dihedral group

- $x \in V$, $E(x) = \{e_1, e_2, e_3\}$
- $s_i = \alpha(l, l, o(e_i), t(e_i))$ edge inversions
- $\langle s_1 \rangle * \langle s_2 \rangle * \langle s_3 \rangle \leq U(\text{id})$
- $g \in U(\text{id})$
  - $\alpha(l, l, x, gx) = s_2 s_1 = g$
  - $U(\text{id}) \leq \langle s_1 \rangle * \langle s_2 \rangle * \langle s_3 \rangle$

color-preserving




Automorphisms of this kind are contained in all universal groups as

$U(\text{id}) \leq U(F)$

# Properties of universal groups

local to global

A universal group $U(F)$ is

- vertex-transitive
- transitive on edges of the same color
- **closed in $\mathbf{Aut}(\boldsymbol{T})$**
  - as such a tdlc Hausdorff group
- compactly generated
- discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1, \ldots, d\}$ is free

# $U(F)$ closed in $\mathrm{Aut}(T)$

Let $g \in \mathrm{Aut}(T) \setminus U(F)$

i.e. $c(g,x) \notin F$ for a $x \in V$.

- $U\big(g, B_1(x)\big) = \{h \in \mathrm{Aut}(T) : h = g \text{ on } B_1(x)\}$
  - open neighborhood of $g$
  - $U\big(g, B_1(x)\big) \subseteq \mathrm{Aut}(T) \setminus U(F)$

$c(g,x) = c(h,x)$
as $gx = hx$

# Properties of universal groups

A universal group $U(F)$ is

- vertex-transitive
- transitive on edges of the same color
- closed in $\mathrm{Aut}(T)$
    - **as such a tdlc Hausdorff group**
- compactly generated
- discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1, \ldots, d\}$ is free

local to global

# Properties of universal groups

local to global

A universal group $U(F)$ is

- vertex-transitive
- transitive on edges of the same color
- closed in $\mathrm{Aut}(T)$
    - as such a tdlc Hausdorff group
- **compactly generated**
- discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1, \dots, d\}$ is free

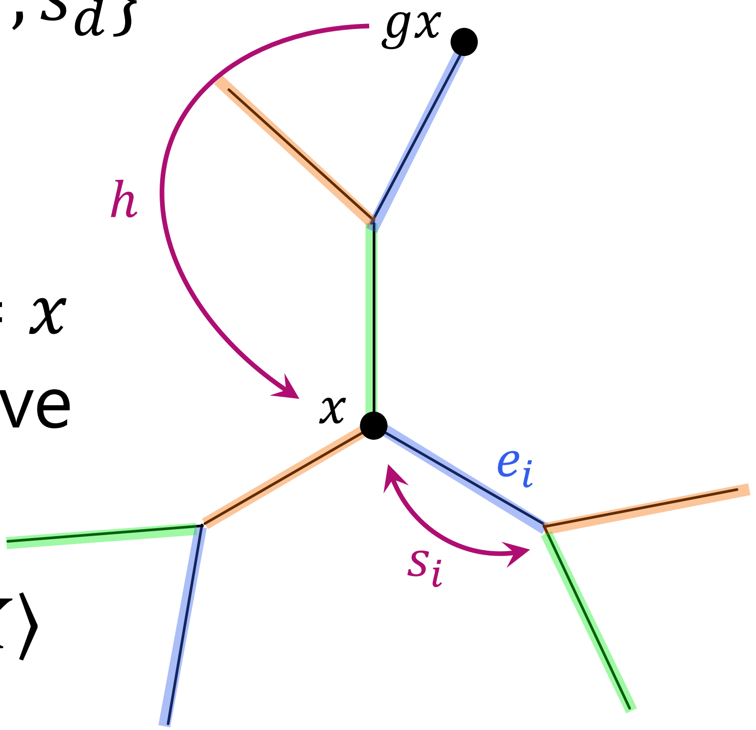
# $U(F)$ compactly generated

Let $x \in V$.

- $\mathrm{St}_{U(F)}(x) = \mathrm{St}(x) \cap U(F) = \{g \in U(F): gx = x\}$
  - compact
- set $K \coloneqq \mathrm{St}_{U(F)}(x) \cup \{s_1, \ldots, s_d\}$
  - compact

Let $g \in U(F)$.

- pick $h \in U(\mathrm{id})$ with $hgx = x$
  - as $U(\mathrm{id})$ vertex-transitive
- $hg \in \mathrm{St}_{U(F)}(x)$
- $g \in U(\mathrm{id}) \cdot \mathrm{St}_{U(F)}(x) = \langle K \rangle$



# Properties of universal groups

local to global

A universal group $U(F)$ is

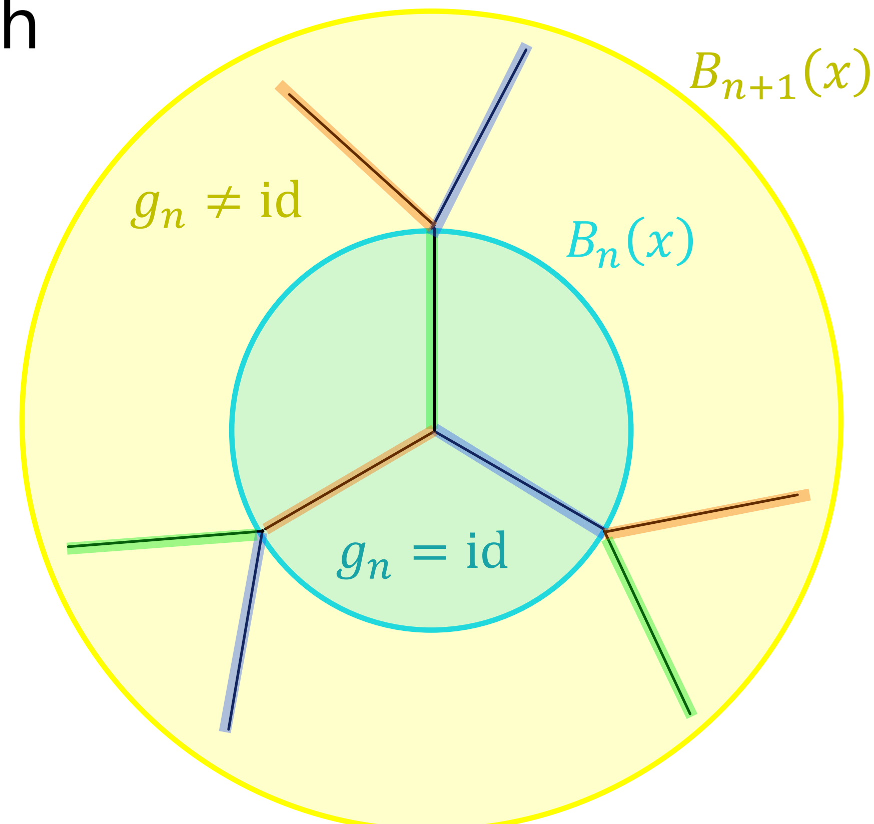
- vertex-transitive
- transitive on edges of the same color
- closed in $\mathrm{Aut}(T)$
  - as such a tdlc Hausdorff group
- compactly generated
- **discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1, \ldots, d\}$ is free**

# $U(F)$ discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1, \ldots, d\}$ free

$U(F)$ *discrete* if $\{\mathrm{id}\}$ is open in $U(F)$

$F \curvearrowright \{1, \ldots, d\}$ not free $\Rightarrow$ $\{\mathrm{id}\}$ not open

- let $x \in V$, $a \in F \setminus \{\mathrm{id}\}$ with $ai = i$
- construct $g_n \in U(F)$ with
  - $g_n = \mathrm{id}$ on $B_n(x)$
  - $g_n \neq \mathrm{id}$ on $B_{n+1}(x)$

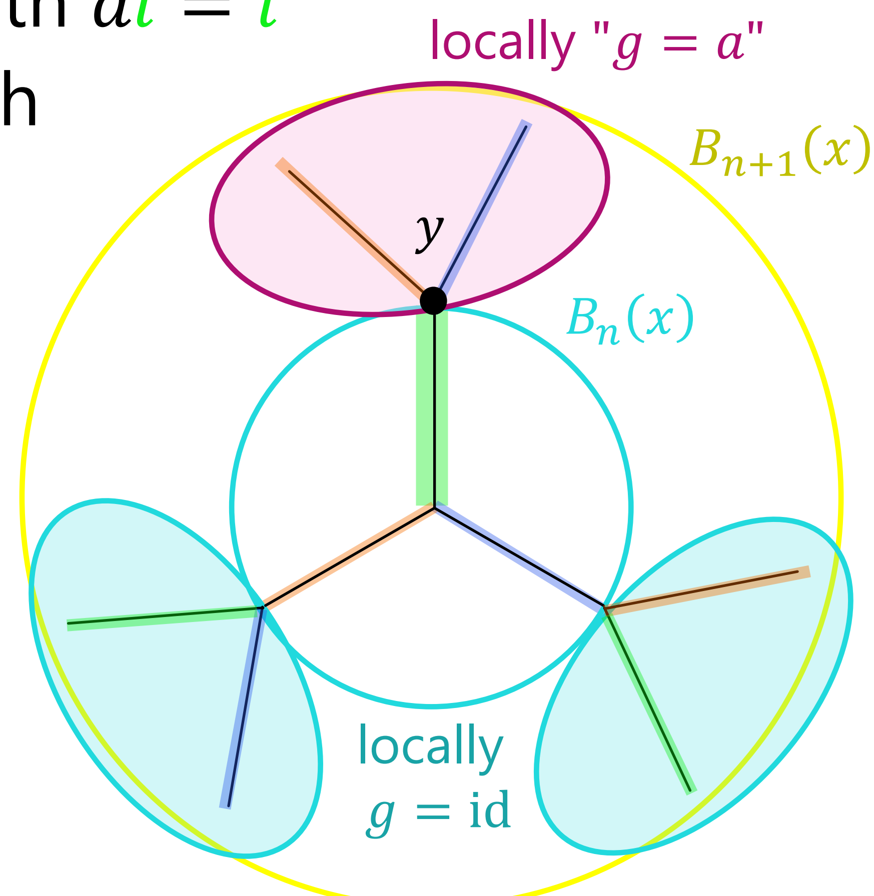


# $U(F)$ discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1, \ldots, d\}$ free

$U(F)$ *discrete* if $\{\mathrm{id}\}$ is open in $U(F)$

$F \curvearrowright \{1, \ldots, d\}$ not free $\Rightarrow \{\mathrm{id}\}$ not open

- let $x \in \mathrm{V}$, $a \in F \setminus \{\mathrm{id}\}$ with $ai = i$
- construct $g_n \in U(F)$ with
  - $g_n = \mathrm{id}$ on $B_n(x)$
  - $g_n \neq \mathrm{id}$ on $B_{n+1}(x)$
- $g_n$ on $B_{n+1}(x)$
  - $g_n|_{E(y)} = l_y^{-1} \circ a \circ l_y$
  - $g_n = \mathrm{id}$ otherwise
- every open neighborhood of id contains some $g_n \neq \mathrm{id}$



# $U(F)$ discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1,\dots,d\}$ free

$U(F)$ *discrete* if $\{\mathrm{id}\}$ is open in $U(F)$.

$g$ as before

<u>$\{\mathrm{id}\}$ not open $\Rightarrow F \curvearrowright \{1,\dots,d\}$ not free</u>

- every open neighborhood of $\{\mathrm{id}\}$ contains a $g_n$ for some $n$
- hence, there is $x \in V$ such that $c(g,x) \in F$ has a fixed point
- $F \curvearrowright \{1,\dots,d\}$ not free

# So far, so good: Properties of $U(F)$

local to global

A universal group $U(F)$ is

- vertex-transitive
- transitive on edges of the same color
- closed in $\mathrm{Aut}(T)$
  - as such a tdlc Hausdorff group
- compactly generated
- discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1, \ldots, d\}$ is free

# So far, so good: Properties of $U(F)$

local to global

A universal group $U(F)$ is

- vertex-transitive
- transitive on edges of the same color
- closed in $\mathrm{Aut}(T)$
  - as such a tdlc Hausdorff group
- compactly generated
- discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1, \dots, d\}$ is free

$\rightsquigarrow$ Cayley-Abels construction

# Big picture: Cayley-Abels graph

$G$ compactly generated tdlc group

$\rightsquigarrow$ Cayley-Abels graph $\Gamma$

$G = U(F)$

$$\mathrm{id} \to \pi_1(\Gamma) \to \mathrm{Aut}(T_d) \to \mathrm{Aut}(G) \to \mathrm{id}$$

$$\begin{array}{ccccccccc} & & & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \end{array}$$

$$\mathrm{id} \to F_d \to \tilde{G} \to G/K \to \mathrm{id}$$

# Big picture: Cayley-Abels graph

$G$ compactly generated tdlc group

⇝ Cayley-Abels graph $\Gamma$

$G = U(F)$

$$
\begin{array}{ccccccccc}
\mathrm{id} \to \pi_1(\Gamma) \to & \mathrm{Aut}(T_d) & \to & \mathrm{Aut}(G) & \to \mathrm{id} \\
& \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & \\
\mathrm{id} \to F_d \to & \tilde{G} & \to & G/K & \to \mathrm{id}
\end{array}
$$

$G$ simple of particular interest

# Big picture: Tits' Simplicity Theorem

Let $G \leq \mathrm{Aut}(T)$ such that

- $G$ geometrically dense and
- $G$ satisfies Tits' Independence Property P

Then the subgroup of "type-preserving" automorphisms

$$G^+ := \langle \mathrm{Fix}_G(e) : e \in E \rangle$$

is either trivial or simple.

# Big picture: Tits' Simplicity Theorem

$G$ moves enough

Let $G \leq \mathrm{Aut}(T)$ such that

- $G$ geometrically dense and
- $G$ satisfies Tits' Independence Property P

Then the subgroup of "type-preserving" automorphisms

$$G^+ := \langle \mathrm{Fix}_G(e) : e \in E \rangle$$

is either trivial or simple.

# Properties of universal groups

A universal group $U(F)$ is

- vertex-transitive
- transitive on edges of the same color
- closed in $\mathrm{Aut}(T)$
  - as such a tdlc Hausdorff group
- compactly generated
- discrete in $\mathrm{Aut}(T)$ iff $F \curvearrowright \{1, \ldots, d\}$ is free

local to global

# Big picture: Tits' Simplicity Theorem

independent outside fixed points

Let $G \leq \mathrm{Aut}(T)$ such that

- $G$ geometrically dense and
- $G$ satisfies Tits' Independence Property P

Then the subgroup of "type-preserving" automorphisms

$$G^+ := \langle \mathrm{Fix}_G(e) : e \in E \rangle$$

is either trivial or simple.

# Open sets ($G = \mathrm{Aut}(T)$)

$\mathrm{Fix}(S) = U(\mathrm{id}, S) = \{h \in G\colon\ hx = x\ \ \forall x \in S\}$

- $U(\mathrm{id}, x) = \{h \in G\colon\ hx = x\}$
- $U(\mathrm{id}, y) = \{h \in G\colon\ hy = y\}$

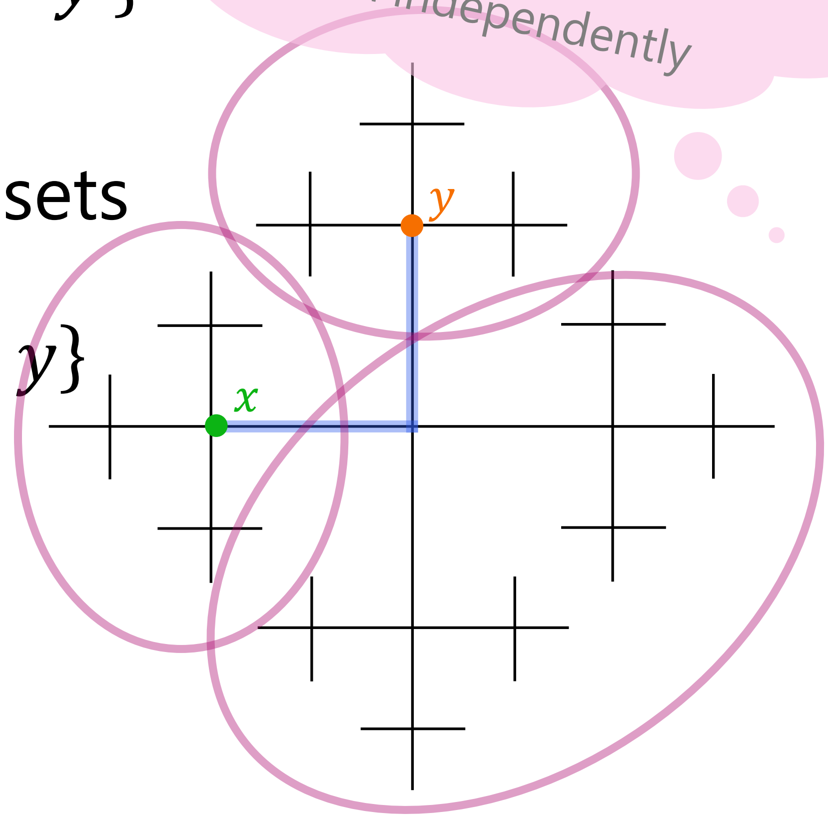
automorphisms outside S can be chosen independently

*finite intersections* of basic sets

$U(\mathrm{id}, x) \cap U(\mathrm{id}, y)$

$= \{h \in G\colon\ hx = x \text{ and } hy = y\}$

$= U(\mathrm{id}, \mathrm{conv}(x, y))$




basic sets with S fixed subtree

# Big picture: Tits' Simplicity Theorem

Let $G \leq \mathrm{Aut}(T)$ such that

- $G$ geometrically dense and
- $G$ satisfies Tits' Independence Property P

Then the subgroup of "type-preserving" automorphisms

$$G^+ := \langle \mathrm{Fix}_G(e) : e \in E \rangle$$

is either trivial or simple.

<u>Corollary</u>

$U(F)^+$ is either trivial or simple.

Sometimes, $[U(F) : U(F)^+] = 2$.

# Big picture: Tits' Simplicity Theorem

Let $G \leq \mathrm{Aut}(T)$ such that

- $G$ geometrically dense and
- $G$ satisfies Tits' Independence Property P

Then the subgroup of "type-preserving" automorphisms

$$G^+ := \langle \mathrm{Fix}_G(e) : e \in E \rangle$$

is either trivial or simple.

Corollary

$U(F)^+$ is either trivial or simple.

Sometimes, $[U(F) : U(F)^+] = 2$.

# Big picture: Cayley-Abels graph

$G$ compactly generated tdlc group

$\rightsquigarrow$ Cayley-Abels graph $\Gamma$

$$\mathrm{id} \to \pi_1(\Gamma) \to \mathrm{Aut}(T_d) \to \mathrm{Aut}(G) \to \mathrm{id}$$

$$\begin{array}{ccc} & \geq & \geq \end{array}$$

$$\mathrm{id} \to F_d \to \tilde{G} \to G/K \to \mathrm{id}$$

$G$ simple of particular interest: $G = U(F)^+$

building blocks of tdlc groups

# What happend today?

trees

tdlc

automorphisms of trees

local to global

tdlc

legal labelings

**universal groups**

Tits' Simplicity

Cayley-Abels

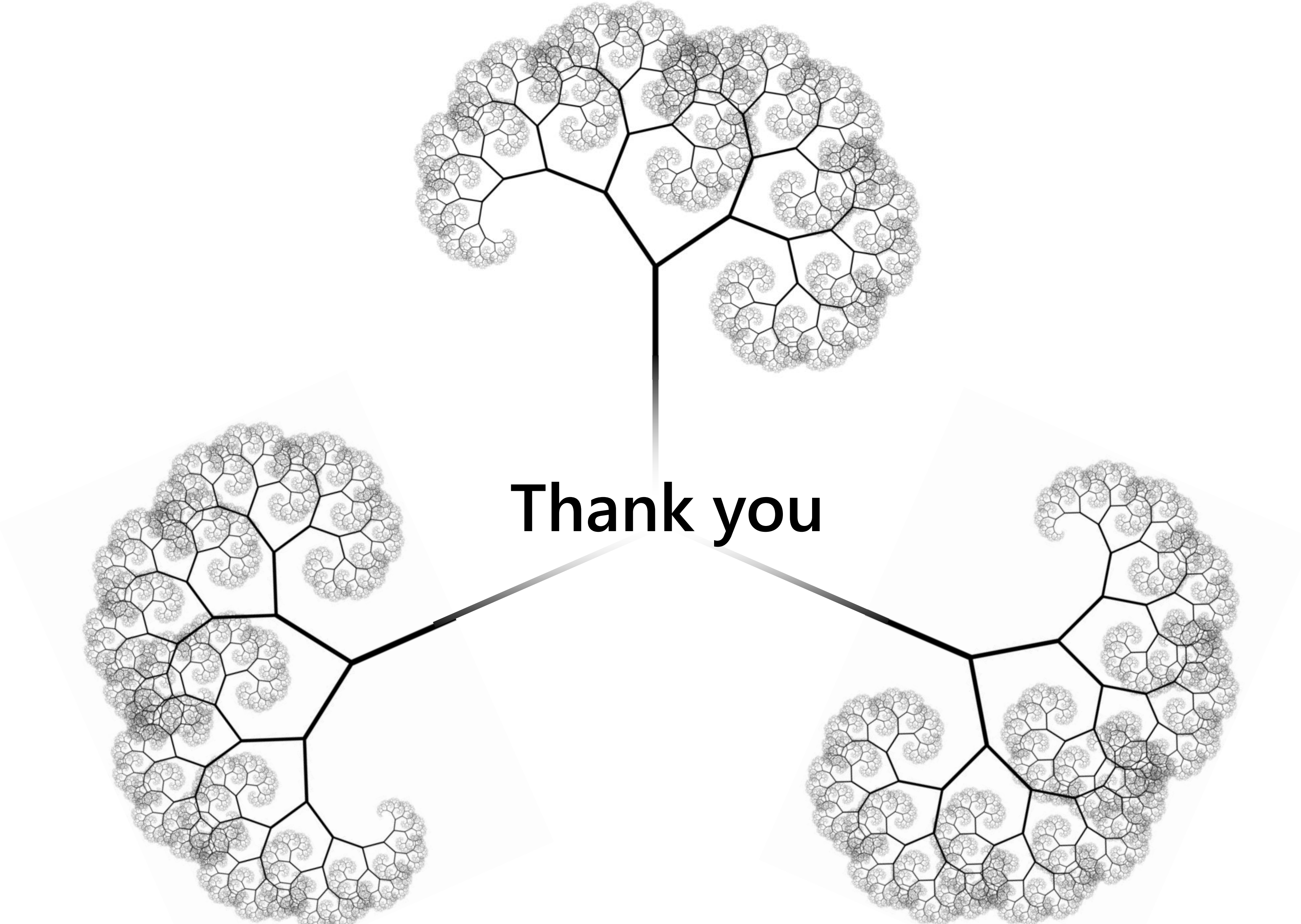
big picture of tdlc groups

## Today's goals

Establish a first intuition for tdlc!

- proof that $Aut(T)$ is tdlc
- draw a legal labeling
- decide if a given universal group is discrete or not

# Thank you



Picture by Marek Fišer

# Sources

- Talk based on
  - *Automorphism groups of trees: generalities and prescribed local actions* (Alejandra Garrido, Yair Glasner and Stephan Tornier) in: *New Directions in Locally Compact Groups* (Edited by Pierre-Emmanuel Caprace and Nicolas Monod)
- picture on first and last page
  - Marek Fišer
  - http://www.marekfiser.com/Projects/AnimatedPythagoras#&gid=1&pid=5