Introduction to tdlc groups

§1 Why (td)lc groups?

§2 Connected vs. td.

§3 Compact generation and Cayley-Abels graphs

§4 Amenability

§ 1 Why (td)lc groups?

(X,τ) lc $:\Longleftrightarrow$ Hausdorff + $\forall x\in X\ \exists K\in \mathcal{U}_x : K$ cpt.

td $:\Longleftrightarrow$ $\forall x \in X :\ x\in \underset{\text{conn.}}{Y} \subseteq X \Rightarrow Y=\{x\}$.

} "tdlc"

Ex. G top. group, G° id. component

G tdlc $\Longleftrightarrow$ $G^\circ = \{e\}$ & $\exists K\in\mathcal{U}_e : K$ cpt.

G discrete $\Longleftarrow$ $\{e\}$ open in G

Why lc groups?

① Many examples (but still "manageable")

② $\exists$ Haar measure (characterizes lc groups) $\longrightarrow$ unitary rep., harmonic analysis

③ (σ-cpt.) $\exists$ coarse structure (for "locally bounded groups") $\longrightarrow$ geometric group theory

Ad ①

(• discrete groups) $\longleftarrow$ nothing to be said using topology

- lc fields: $\underbrace{\mathbb{R},\mathbb{C}}_{\text{Archimedean}}$, $\underbrace{\mathbb{Q}_p \text{ \& finite ext.},\ \mathbb{F}_p((t)) \text{ \& finite ext.}}_{\text{non-Archimedean}}$
- matrix groups over such fields, e.g. $SL_n(\mathbb{K})$, $Sp_{2n}(\mathbb{K})$, $O(\mathbb{K}^n, q)$, ... $\nabla(\mathbb{K})$, ...

- (X,d) metric space, proper ($\bar{B}_r(x)$ compact)

  $\rightsquigarrow$ $\mathrm{Is}(X,d)$ is an lc group w.r.t. pointwise convergence (+ second countable)
  s.c.
- $\mathrm{Aut}(T)$, T a tree, is a special case

Theorem (Malicki/Solecki '09)
Every lcsc group G is isomorphic to $\mathrm{Is}(X,d)$ for some (X,d).

- restricted products / projective limits

  (G_i) lcp., $K_i \leq G_i$ cpt. $\Longrightarrow$ $\underbrace{\prod{}'(G_i,K_i) = \{(g_i)\in \prod G_i \mid g_i\in K_i \text{ for almost all } i\}}_{\text{lcp group}}$

  ("∞-dim")

Ad ② $\exists\, m_G$ left-inv. Radon measure on G, unique up to scaling

Ad ③ A left-invariant pseudo metric $d : G \times G \to [0,\infty)$ is *adapted* if it is bounded on compact subsets and proper.

Lemma

G σ-compact lc $\Rightarrow$ $\exists$ adapted d

Proof: $G = \bigcup_{n=1}^{\infty} S_n$, $S_1 \subseteq S_2 \subseteq \ldots$ cpt.

$$d_n(g,h) := \begin{cases} |g^{-1}h|_{S_n} & ; \ g^{-1}h \in \langle S_n \rangle \\ \infty & , \ \text{else} \end{cases}$$

$$\rightsquigarrow \ d(g,h) := \inf \left\{ \sum_{i=0}^{k-1} j_i \, d_{j_i}(g_i, g_{i+1}) \ \middle| \ g = g_0, g_1, \ldots, g_k = h \in G \right\}$$

Metric Coarse category: • objects are *metric spaces* (X, d_X), $(Y, d_Y), \ldots$

• $f : X \to Y$ *coarsely lipschitz* if

$$d_Y(f(x_1), f(x_2)) \le \underbrace{\Phi_+}_{\text{non-decreasing}}(d_X(x_1,x_2)).$$

• $f, f' : X \to Y$ are *close* if $\sup_{x \in X} d_Y(f(x), f'(x)) < \infty$.

• *morphisms* are closeness classes of Coarse Lipschitz maps

Exercise d_1, d_2 adapted $\Rightarrow$ $(G, d_1) \cong (G, d_2)$

$\rightarrow$ $[G]_c$ isomorphism class of (G,d) (*Coarse class* of G)

How to find models for $[G]_c$?

- For *Lie groups*: d left-invariant Riem. metric $\rightarrow$ (diff-geo.)
- If *compactly-generated*: Cayley graph / Cayley-Abels graph (see below)
- In general: Via "*geometric actions*"

$G \overset{\text{isom.}}{\curvearrowright} (X, d_X)$ is <u>cobounded</u> $\iff$ $\exists\, Y \subset X$ bounded: $G.Y = X$.

<u>loc. bounded</u> $\iff$ $K \subset G$ cpt. $\Rightarrow$ $K.x_0 \subset X$ bounded

<u>metrically proper</u> $\iff$ $\forall R > 0, x_0 \in X$: $\{ g \in G \mid d(g.x_0, x_0) \le R \}$ cpt.

<u>geometric</u> $\Longleftarrow$ all

<u>Rem.</u> $G \curvearrowright (X, d_X)$ geom

$$\Longrightarrow \quad d(g,h) := d_X(g.x_0, h.x_0) \quad \text{adapted pseudo-metric}$$

$$\Rightarrow \quad (G, d) \xrightarrow{\cong} (X, d_X), \quad g \mapsto g.x_0$$

$$\Rightarrow \quad (X, d_X) \in [G]_c.$$

<u>Outlook</u> ·) G compactly gen. tdlc

$$\Rightarrow \exists \text{ loc. finite regular graph } \mathcal{G} \in [G]_c.$$

→ <u>"combinatorics"</u>

·) G simple alg. group $/k$ → $G \overset{\text{geom.}}{\curvearrowright}$ building (sym. space)

§2 Connected vs. td

G td. group $\rightsquigarrow$ G° connected td. group

$\rightsquigarrow$ all examples are "essentially algebraic":

Theorem (Gleason–Yamabe)

$\forall U \in \mathcal{U}_e^{G^\circ}\ \exists \underset{\text{cpt}}{K} \leq U:\ K \trianglelefteq G^\circ$ and G°/K Lie group (~ matrix groups over $\mathbb{R}, \mathbb{C}$)

$\rightsquigarrow$ connected lc groups are "understood", what is left?

Lemma

$L := G/G^\circ$ is tdlc

Proof: $\pi: G \twoheadrightarrow L$ Claim: $\pi^{-1}(L^\circ)$ is connected!

(Indeed, assume $\pi^{-1}(L^\circ) = F_1 \sqcup F_2$, F_i clopen.

$G^\circ \subset \pi^{-1}(L^\circ) \Rightarrow \forall g \in \pi^{-1}(L^\circ):\ \underbrace{gG^\circ}_{\text{con.}} \subset F_1$ or $\underbrace{gG^\circ}_{\text{con.}} \subset F_2$

$\Rightarrow L^\circ = \pi(F_1) \sqcup \pi(F_2) \Rightarrow F_1 = \emptyset$ or $F_2 = \emptyset$.)

But then $\pi^{-1}(L^\circ) = G^\circ \Rightarrow L^\circ = \{e\}$.

~~How do tdlc spaces look like~~

⚠ Not every tdlc group is essentially algebraic!

There are new generic examples. $\rightarrow$ $\boxed{\text{Goal of this semester}}$
(not just Lie groups / sym. spaces)

How do tdlc spaces look like?

Lemma X cpt. $\Rightarrow \forall x \in X:\ K_x := \bigcap \{U \mid x \in U,\ U \text{ clopen}\}$ is connected

Proof: Assume $K_x = \underset{\underset{x}{\uparrow}}{K_1} \sqcup K_2$, K_i clopen. $\Rightarrow \exists U_1, U_2$ open: $U_i \supseteq K_i$, $U_1 \cap U_2 = \emptyset$

Set $F := \overline{U_2} \setminus U_2$, then $F \cap K_2 = \emptyset$ and $F \cap K_1 = \emptyset$, hence $F \cap K_x = \emptyset$.

$\overset{\text{compactness}}{\Longrightarrow} \exists U_i$ clopen: $\underbrace{\left(\bigcap_{i=1}^{n} U_i\right)}_{=:V} \cap F = \emptyset$

$\Rightarrow V' := \underset{\underset{x}{\uparrow}}{V} \cap U_2^c$ clopen, $V' \cap K_2 = \emptyset \Rightarrow K_2 = \emptyset$. ■

Cor. X tdlc, $x \in X \Rightarrow x$ has basis of compact open nbds.

Proof: By lemma, $\{x\} = \bigcap \{U \mid \cancel{\text{clopen}},\ U \overset{\text{clopen}}{\subseteq} C,\ x \in U\}$ for any compact nbd. $C \ni x$.

Now $\underbrace{(C \setminus \mathring{C})}_{\text{cpt.}} \cap \{x\} = \emptyset \Longrightarrow (C \setminus \mathring{C}) \cap \bigcap_{i=1}^{n} \underbrace{U_i}_{\text{clopen}} = \emptyset$

$\Rightarrow x \in \bigcap_{i=1}^{n} U_i \overset{\text{clopen}}{\subseteq} X.$ ∎

Thm. (van Dantzig)

G tdlc group $\Longrightarrow \mathcal{U}_e$ has a basis of compact open subgroups.

Proof: We need

Lemma G top. group, $\overset{(\text{cpt.})}{C} \subset \overset{(\text{clopen})}{C'} \subset G \Rightarrow \exists \underset{V^{-1}}{\overset{\shortparallel}{V}} \in \mathcal{U}_e : C V \subset C'$

Let $U \in \mathcal{U}_e$; by the cor. $\exists\, e \in \underbrace{C \subset U}$ compact open in G.

By lemma, $\exists V \in \mathcal{U}_e : C \cdot V = C$. (Choose $C' := C$)

Set $L := \{g \in G \mid C \cdot g = C\} \supseteq V$. Then L is open (since it contains V) and $L \subseteq C$ is closed, hence compact. ∎

Cor. G compact tdlc $\Longleftrightarrow G \overset{\text{closed}}{<} \prod_{F \in \mathcal{F}} F$, $|F| < \infty$ ("pro-finite")

Proof: $H < G$ compact-open $\overset{G \text{ compact}}{\Longrightarrow} |G/H| < \infty$

$\Rightarrow \bigcap_{g \in G/H} g H g^{-1} \lhd G$ compact open

$\Rightarrow G$ has basis of normal compact open subgroups (H_i)

$\Rightarrow G \hookrightarrow \prod \underbrace{G/H_i}_{\text{finite grps.}}$. ∎

§ 3 Compact generation and Cayley-Abels graphs

Proposition

G σ-cpt. l.c. group. TFAE

(i) G is compactly generated

(ii) G acts geometrically on a geodesic metric space (X, d_X)

(iii) G admits an adapted coarsely connected pseudo-metric d.

↳ $\exists C\ \forall g,h\ \exists\ g = g_0, \dots, g_k = h \in G : d(g_i, g_{i+1}) < C$.

Proof: (i) $\Rightarrow$ (ii): $G = \langle S \rangle$ (S cpt.) $\rightsquigarrow$ $G \overset{\text{geo.}}{\curvearrowright} \mathrm{Cay}(G,S)$.

(ii) $\Rightarrow$ (iii): $G \overset{\text{geo.}}{\curvearrowright} (X, d_X)$ $\rightsquigarrow$ $d(g,h) := d_X(g.x_0, h.x_0)$.

d_X geod. $\Rightarrow$ d coarsely connected

(iii) $\Rightarrow$ (i): Let $G_n := \langle B(e,n) \rangle$, then $d(h_1 G_n, h_2 G_n) > n$ if $h_1 G_n \neq h_2 G_n$.

Now if C is as in (iii) and $n > C$, then no C-path connects different cosets.

Thus $G_n = G$. ∎

The Cayley graph is a (connected) model for $[G]_c$, but locally infinite.

Better:

Construction (Abels)

G (compactly generated) tdlc group, S compact generating set + $H < G$ compact open subgroup ("arbitrary small")

$$\mathcal{G} = \big(\underbrace{G/H}_{\text{(discrete!)}},\ \{(gH, gcH) \mid g \in G,\ c \in HSH\}\big)$$

Cayley-Abels graph

$G \curvearrowright \mathcal{G}$ vertex transitive, $|v(x)| = |v(e)| =: d = |\underbrace{HSH}_{\text{cpl.}}/\underbrace{H}_{\text{open}}| < \infty$

will lead $K := \bigcap_{g \in G} g H g^{-1}$.

Cor. G (compactly generated) tdlc $\Rightarrow$ $[G]_c \ni \mathcal{G}$ locally finite d-regular graph, $G \curvearrowright \mathcal{G}$ continuously ∎

Note: G c.ge. tdlc. $\rightsquigarrow$ $K \lhd G$ arbitrary small:

$$G/K \hookrightarrow \mathrm{Aut}(\mathcal{G})$$

$$\Rightarrow \quad \begin{tikzcd} \{e\} \arrow[r] & \pi_1(\mathcal{G}) \arrow[r] \arrow[d, equal] & \mathrm{Aut}(T_d) \arrow[r] & \mathrm{Aut}(\mathcal{G}) \arrow[r] & \{e\} \\ \{e\} \arrow[r] & \underbrace{\pi_1(\mathcal{G})}_{\text{free group}} \arrow[r] & \tilde{G}_K \arrow[u, phantom, "\geq" rotate=90] \arrow[r] & G/K \arrow[u, phantom, "\geq" rotate=90] & \{e\} \end{tikzcd}$$

Cor. G c.ge. tdlc $\Longrightarrow$ $\forall U \in \mathcal{U}_e \ \exists K \subset U,\ K \trianglelefteq G$,
$\exists \tilde{G}_K \leq \mathrm{Aut}(T_d)$: $\tilde{G}_K \twoheadrightarrow G/K$
with discrete kernel (

$\rightarrow$ Study subgroups of $\mathrm{Aut}(T_d)$!

Cor. G c.ge. tdlc <u>top. simple</u> $\Longrightarrow$ $\exists$ free group F (countable):

$$F \rightarrow \tilde{G} \rightarrow G$$
$$\tilde{G} \leq \mathrm{Aut}(T_d)$$

§4 Amenability

G amenable $\iff$ $\left(G \overset{\text{linear}}{\curvearrowright} \underset{\substack{\text{compact} \\ \text{convex}}}{S} \subseteq E \text{ lcv. tvs.} \Rightarrow S^G \neq \emptyset \right)$
(lc)

Ex. $G = SL_n(\mathbb{R})$

$K = SO_n(\mathbb{R})$, $P = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$ $\Rightarrow$ $G = KP$ (Gram-Schmidt)

$\Rightarrow [G]_c = [P]_c$

$SL_n(\mathbb{R}) \overset{\text{closed}}{>} SL_2(\mathbb{R}) \overset{\text{discrete}}{>} F_2 \Rightarrow SL_n(\mathbb{R})$ not amenable

P soluble $\Rightarrow$ P amenable

Cor. Amenability is not a geometric property! ∎

Rem. Interestingly, "amenable + unimodular" is a geometric property!
(Tessera)

Theorem (Amenable radical, Burger-Monod)
G lc group $\Rightarrow \exists!$ max closed amenable normal subgroup $A(G) \trianglelefteq G$.
($A(G)$ is top. char. and $A(G/A(G)) = 1$.)

Example
G connected $\Rightarrow$ $\underbrace{G/K}_{H}$ Lie $\Rightarrow$ $A(G) \supseteq K$

$H = (\underbrace{H_1 \times \dots \times H_e}_{\text{non-compact simple}} \times \underbrace{K_1 \times \dots \times K_e}_{\text{compact}}) \ltimes \underbrace{S}_{\text{soluble}}$

$\Rightarrow G/A(G) = H_1 \times \dots \times H_e$ semisimple Lie group

$\boxed{G \text{ tdlc} \rightsquigarrow G/A(G) = ???}$

Proof of Theorem

Set $A(G) := \langle N \lhd G \mid N \text{ closed, amenable} \rangle$ and let S be a compact-convex $A(G)$-space. Let $N_1, \dots, N_r \lhd G$ closed, amenable.

Then $N_1 \cdots N_{r-1} \lhd G \Rightarrow G \curvearrowright S^{N_1 \cdots N_{r-1}} \Rightarrow N_r \curvearrowright S^{N_1 \cdots N_{r-1}}$

$$\Rightarrow \left(S^{N_1 \cdots N_{r-1}}\right)^{N_r} = \bigcap_{i=1}^{r} S^{N_i}$$

Inductively we thus see that $S^{N_1} \neq \emptyset$ (N_1 amenable)

$$\Rightarrow (S^{N_1})^{N_2} = S^{N_1 N_2} \neq \emptyset \quad (N_2 \text{ amenable})$$

$$\Rightarrow \dots \Rightarrow S^{N_1 \cdots N_r} \neq \emptyset$$

S compact, $\bigcap_{i=1}^{r} S^{N_i} \neq \emptyset \Longrightarrow \bigcap_{N} S^{N} \neq \emptyset$

$$\xrightarrow{\text{continuity}} S^{A(G)} \neq \emptyset.$$

Thus $A(G)$ amenable, rest follows. ∎