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# Algebra II – Problem Sheet 1

By convention every ring on every problem sheet is commutative and has an identity element 1.

**Exercise 1** (4 points)
Let $R$ be an integral ring and $M$ be an $R$-module. The set of torsion elements of $M$ is

$$T(M) := \{x \in M \mid \exists\, a \in R \setminus \{0\} : \; ax = 0\}.$$

$M$ is called *torsion-free* if $T(M) = \{0\}$.

a) Show that $T(M)$ is a submodule of $M$.

b) Show that $M$ is torsion-free if $M$ is free.

c) Give an example for an $R$-module $M$ which is torsion-free but not free. (Hint: Is $R = \mathbb{Z}$ possible?)

**Exercise 2** (4 points)
Given a ring $R$ and an $R$-module $M$, the *dual module* is defined as

$$M^* := \mathrm{Hom}_R(M, R).$$

a) Show that if $M$ is free and finitely generated then $M^*$ is free. (Hint: Remember what a *dual basis* is.)

b) Show that $M^*$ is torsion-free (compare Exercise 1) if $R$ is integral.

c) Determine the dual modules of the following $\mathbb{Z}$-modules:

$$\mathbb{Z}, \quad n\mathbb{Z}, \quad \mathbb{Z}/n\mathbb{Z}, \quad \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}, \quad \mathbb{Q} \qquad (n \geq 2)$$

**Exercise 3** (4 points)
Let $R, S$ be rings and $\varphi : R \to S$ a ring homomorphism.

a) Find a structure on $S$ as an $R$-module which is compatible with the structure on $S$ as a ring, that means, for all $a, b \in R$ and all $x, y \in S$ the equation

$$(ax) \cdot (by) = (a \cdot b)(x \cdot y)$$

holds.

b) Show that every $S$-module $M$ is also an $R$-module.

c) Is $M$ free as $R$-module if it is free as $S$-module? Conversely, is $M$ free as $S$-module if it is free as $R$-module?

**Exercise 4** (4 points)
Let $R$ be a principal ideal domain (in particular, $R$ is integral) and $M$ be an $R$-module. $M$ is called *divisible* if for all $x \in M$ and all $a \in R \setminus \{0\}$ there is a $y \in M$ such that $ay = x$.

a) Show that $M$ is divisible if and only if $M$ is injective. (Hint: Use Zorn's lemma.)

b) Show that for an injective $R$-module $M$, every quotient module $M/U$ is also injective.

**Solutions to be handed in** on Tuesday, 22. 4. 2008, at the beginning of the problem session in S12.