Universität Karlsruhe
Institut für Algebra und Geometrie
Prof. Dr. F. Herrlich
Ute Wolf

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# Algebra II – Problem Sheet 2

**Exercise 1** (4 points)
Determine the tensor product of each two of the following $\mathbb{Z}$-modules:

$$\mathbb{Z}/n\mathbb{Z}, \quad \mathbb{Z}/m\mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{Q}/\mathbb{Z} \qquad (n, m \geq 2)$$

**Exercise 2** (4 points)
Let $M, N, P$ be $R$-modules. Show that

$$\mathrm{Hom}(M \otimes_R N, P) \cong \mathrm{Hom}(M, \mathrm{Hom}(N, P)).$$

**Exercise 3** (4 points)

a) Let $R$ be a ring. Determine all free submodules of $R$.

Now let $R := \mathbb{Z}[\sqrt{-5}]$ and $I := (2, 1 + \sqrt{-5})$.

b) Show that $I$ is not free.

c) Show that $I$ is projective. (Hint: Use the homomorphism $\Phi : I \to R^2$ that maps 2 to $(1 - 3\sqrt{-5}, 5 + \sqrt{-5})$ and $1 + \sqrt{-5}$ to $(8 - \sqrt{-5}, 3\sqrt{-5})$.)

**Exercise 4** (4 points)

a) Show that every subgroup $A$ of a free abelian group $F$ is free. Use without proof that every basis $B$ of $F$ can be well-ordered, that means, it can be totally ordered such that every nonempty subset of $B$ has a least element. (Hint: For every $b_j \in B$, consider $F_j := \langle \{b' \in B \,:\, b' \leq b_j\} \rangle$ and $p_j : F_j \cap A \to \mathbb{Z}$, $\sum\limits_{b' \leq b_j} \lambda_{b'} b' \mapsto \lambda_{b_j}$. There is some $u_j \in F_j \cap A$ such that $p_j(u_j)$ generates $\mathrm{im}(p_j)$.)

b) Show that every projective abelian group is free.

**Solutions to be handed in** on Tuesday, 29. 4. 2008, at the beginning of the problem session in S12.