Universität Karlsruhe
Institut für Algebra und Geometrie
Prof. Dr. F. Herrlich
Ute Wolf

29. 4. 2008

Algebra II – Problem Sheet 3

Exercise 1 (4 points)
Let R be a ring, $S \subseteq R$ be a multiplicatively closed subset of R such that $1 \in S$, and M be an R-module. On $M \times S$ we define an equivalence relation by

$$(m, s) \sim (m', s') \;:\Leftrightarrow\; \exists\, t \in S: \; t \cdot (sm' - s'm) = 0.$$

We denote the equivalence class of (m, s) by $\frac{m}{s}$. The set $(M \times S)/\sim$ is called *localization* of M at S and is denoted by $S^{-1}M$ or M_S. The following operations turn M_S into an R-module ($m, m' \in M$, $s, s' \in S$, $r \in R$):

$$\begin{aligned} \tfrac{m}{s} + \tfrac{m'}{s'} &:= \tfrac{s'm + sm'}{ss'} \\ r \cdot \tfrac{m}{s} &:= \tfrac{rm}{s} \end{aligned}$$

a) Show that $M_S \cong M \otimes_R R_S$.

b) Show that, for any R_S-module U, the modules $M \otimes_R U$ and $M_S \otimes_{R_S} U$ are isomorphic.

c) Now let N be another R-module. Show that $M_S \otimes_{R_S} N_S \cong (M \otimes_R N)_S$.

Exercise 2 (4 points)
In Exercise 1, we defined localization. Similarly to the case of localization of rings, for an R-module M and a prime ideal $P \trianglelefteq R$ we denote the localization of M at $R \setminus P$ by M_P. Show that if an R-module M is flat, then for all prime ideals $P \trianglelefteq R$, M_P is flat as an R_P-module.

(*Remark:* The other direction is also true.)

Exercise 3 (4 points)
Let R be a ring and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_2} & & \big\downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \longrightarrow & 0 \end{array}$$

be a commutative diagram of R-modules with exact rows. Show that f_2 is injective if f_1 and f_3 are injective. Show that f_2 is surjective if f_1 and f_3 are surjective.

Exercise 4 (4 points)
Let R be a ring and $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of R-modules where C is flat. Show that:

a) For any R-module M, the induced sequence

$$0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

is exact. (Hint: Fit all exact sequences you can imagine into one big diagram and chase elements around.)

b) A is flat if and only if B is flat.

Solutions to be handed in on Tuesday, 6. 5. 2008, at the beginning of the problem session in S12.