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Algebra II – Problem Sheet 4

Exercise 1 (4 points)
Let $d \in \mathbb{N}$. Show that the assignment

$$M \;\mapsto\; \wedge^d M$$

is a covariant functor from the category of R-modules into itself.

Exercise 2 (4 points)
Let R be a ring in which 2 is a unit. Show that for any $d \geq 2$ and any ideal $I \trianglelefteq R$ the following equations hold:

a) $I \cdot \wedge^d(I) = 0$

b) $\wedge^d(I) \cong \wedge^d(I/I^2)$

Exercise 3 (4 points)
Let A be a finitely generated free abelian group and $n \geq 1$ be an integer.

a) Show that the assignment

$$x_1 \wedge \ldots \wedge x_n \;\mapsto\; \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \cdot x_{\pi(1)} \otimes \ldots \otimes x_{\pi(n)}$$

defines an injective homomorphism $\Phi_n : \wedge^n(A) \to T^n(A)$.

b) Determine $p_n \circ \Phi_n$, where $p_n : T^n(A) \to \wedge^n(A)$ is the canonical projection.

c) Find an injective homomorphism $\Psi : \wedge^3(A) \to \wedge^2(A) \otimes A$.

Exercise 4 (4 points)
Let k be a field and $n \geq d \geq 1$ integers. On $\wedge^d k^n \setminus \{0\}$ we define an equivalence relation by $x \sim y \;:\Leftrightarrow\; \exists \lambda \in k^\times : \; y = \lambda x$. We set $\mathbb{P}(\wedge^d k^n) := (\wedge^d k^n \setminus \{0\})/\!\sim$. Finally, let $G(d,n)$ be the set of all d-dimensional subspaces of k^n.

a) Show that, for any $U \in G(d,n)$ and any basis $B = \{u_1, \dots, u_d\}$ of U, the element $[u_1 \wedge \ldots \wedge u_d]_\sim \in \mathbb{P}(\wedge^d k^n)$ does not depend on the choice of B, and hence the map $f_{d,n} : G(d,n) \to \mathbb{P}(\wedge^d k^n)$ which sends U to $[u_1 \wedge \ldots \wedge u_d]_\sim$ is well-defined.

b) Show that $f_{d,n}$ is injective.

c) Determine the image of $f_{1,n}$ in $\mathbb{P}(\wedge^1 k^n)$.

d) Now let $d = 2$ and $n = 4$. For $y \in \wedge^2 k^4 \setminus \{0\}$, $y = \sum\limits_{1 \leq i < j \leq 4} a_{ij} b_i \wedge b_j$, show that $[y]_\sim$ is in the image of $f_{2,4}$ if and only if the linear map

$$\varphi_y : k^4 \longrightarrow \wedge^3 k^4, \;\; x \mapsto y \wedge x.$$

has rank 2.

Solutions to be handed in on Tuesday, 13. 5. 2008, at the beginning of the problem session in S12.