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Algebra II – Problem Sheet 8

Exercise 1 (4 points)
Let $S = \bigoplus_{i \in \mathbb{N}} S_i$ be a graded ring.

a) For an ideal $I \trianglelefteq S$, let $I_{\text{hom}} := (\{x \in I \,:\, x \text{ homogeneous}\})$ be the subideal generated by all homogeneous elements of I. Show that I_{hom} is prime if I is prime. Is the converse also true?

b) Let I be a homogeneous ideal and let $V(I)$ be the set of all prime ideals of S containing I. Show that any minimal element of $V(I)$ is homogeneous.

Exercise 2 (6 points)
Let R be a ring, $I \trianglelefteq R$ be an ideal and M be an R-module. We define

$$\mathrm{gr}_I(M) := \bigoplus_{n \in \mathbb{N}} I^n M / I^{n+1} M.$$

Show that:

a) $\mathrm{gr}_I(R)$ is a graded ring.

b) $\mathrm{gr}_I(M)$ is a graded $\mathrm{gr}_I(R)$-module.

c) $\mathrm{gr}_I(R)$ is noetherian if R is noetherian.

d) If $R = \mathbb{Z}$ and $I = p\mathbb{Z}$ for a prime number p then $\mathrm{gr}_I(R) \cong \mathbb{F}_p[X]$.

Exercise 3 (6 points)
Let K be a field, $S = K[X, Y, Z]$ and $f \in S$ be a homogeneous polynomial of degree d.

a) Determine the Hilbert polynomial of $S/(f)$.

b) Determine the Hilbert series of S.

Solutions to be handed in on Tuesday, 10. 6. 2008, at the beginning of the problem session in S12.