Universität Karlsruhe 10. 6. 2008
Institut für Algebra und Geometrie
Prof. Dr. F. Herrlich
Ute Wolf

Algebra II – Problem Sheet 9

Exercise 1 (4 points)
Let K be a field with $\mathrm{char}(K) \neq 2$. Let $\sigma, \tau \in \mathrm{Aut}_K(K[X,Y])$ be the automorphisms given by $\sigma(X) = -X$, $\sigma(Y) = Y$, $\tau(X) = X$, $\tau(Y) = -Y$ and let G be the subgroup of $\mathrm{Aut}_K(K[X,Y])$ generated by σ and τ. Determine generators of the G-invariants $K[X,Y]^G$ in $K[X,Y]$.

Exercise 2 (6 points)
Let K be an algebraically closed field with $\mathrm{char}(K) = 0$ and $G \leq \mathrm{GL}_n(K)$ be a finite subgroup of $\mathrm{GL}_n(K)$.

a) Let G act linearly on a finite-dimensional K-vector space V. Show that

$$\dim(V^G) = \frac{1}{|G|} \cdot \sum_{g \in G} \mathrm{tr}(g).$$

b) Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $g \in G$. Show that the endomorphism g_d of $K[X_1, \ldots, X_n]_d$ induced by g has the trace

$$\mathrm{tr}(g_d) = \sum_{d_1 + \ldots + d_n = d} \lambda_1^{d_1} \ldots \lambda_n^{d_n} .$$

c) Show that the Hilbert series of $K[X_1, \ldots, X_n]^G$ ist given by

$$H(t) = \frac{1}{|G|} \cdot \sum_{g \in G} \det(1 - tg)^{-1}.$$

Exercise 3 (4 points)
Let R be a ring and $I \trianglelefteq R$ be an ideal.

a) Show that $\sqrt{I}$ is the intersection of all prime ideals containing I.

   Hint: For an element $x \in R \setminus \sqrt{I}$, consider the set $\mathcal{M}$ of all ideals $J \supseteq I$ having empty intersection with $\{x^n : n \in \mathbb{N}\}$.

b) Show that, if R is noetherian, there are finitely many prime ideals $P_1, \ldots, P_m$ of R such that $\sqrt{I} = P_1 \cap \ldots \cap P_m$.

Exercise 4 (2 points)
Let R be a ring, $I \trianglelefteq R$ be a nilpotent ideal in R (i.e. there is an $n \in \mathbb{N}$ such that $I^n = 0$) and M be an arbitrary (not necessarily finitely generated) R-module. Show that $IM = M$ implies $M = 0$.

Solutions to be handed in on Tuesday, 17. 6. 2008, at the beginning of the problem session in S12.