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Algebra II – Problem Sheet 10

Exercise 1 (4 points)
Let R be a local ring with maximal ideal m and M, N be finitely generated R-modules.

a) Show that $M = 0 \Leftrightarrow M/mM = 0$.

b) If $M \neq 0$, show that there is a surjective homomorphism $M \otimes_R R/m \to R/m$ and hence a surjective homomorphism $M \otimes_R N \to N/mN$.

c) Show that if $M \otimes_R N = 0$ then $M = 0$ or $N = 0$.

Exercise 2 (4 points)
Let $R := \{f = \sum_{i=0}^{d} a_i X^i \in \mathbb{Q}[X] \,:\, a_0 \in \mathbb{Z}\}$.

a) Show that R is a ring which is not noetherian.

b) Find an ideal $I \trianglelefteq R$, $I \neq R$ such that $\bigcap_{n \in \mathbb{N}} I^n \neq 0$.

Exercise 3 (4 points)
Let R be a noetherian local ring and m be its maximal ideal.

a) Show that if there is an $n \in \mathbb{N}$ such that $m^n = m^{n+1}$, then $\dim(R) = 0$.

b) Find an example for a noetherian local ring with maximal ideal $m \neq 0$ in which the assumption in a) is fulfilled.

Exercise 4 (4 points)
Let K be a field and $R := K[X_1, X_2, \ldots]$ be the polynomial ring in countably many variables. We define the following prime ideals in R:

$$\begin{aligned} P_0 &:= (X_1), \\ P_1 &:= (X_2), \\ P_2 &:= (X_3, X_4), \\ P_3 &:= (X_5, \ldots, X_8), \\ &\vdots \\ P_n &:= (X_{2^{n-1}+1}, \ldots, X_{2^n}) \qquad (n \in \mathbb{N}_{\geq 1}) . \end{aligned}$$

Let $U := R \setminus \bigcup_{n \in \mathbb{N}} P_n$ and $S := R_U$.

a) Determine all maximal ideals of S.

b) Show that S is noetherian.

Hint: Use the fact that $S_{P_n S} \cong R_{P_n}$. Apply Exercise 3 on Problem Sheet 6.

c) Show that $\dim(S) = \infty$.

Solutions to be handed in on Tuesday, 24. 6. 2008, at the beginning of the problem session in S12.