Universität Karlsruhe 1. 7. 2008
Institut für Algebra und Geometrie
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Algebra II – Problem Sheet 12

Exercise 1 (5 points)
Let R be a local ring with maximal ideal m. For a finitely generated R-module M, let $\mu(M)$ be the minimal number of generators.

a) Show that $\mu(M) = \dim_{R/m}(M/mM)$.

b) Show that in any system of generators of M there are $\mu(M)$ elements generating M.

Now let R be a local, finitely generated, integral algebra over a field K with maximal ideal m. For any element $x \in m$, let $P_x \subseteq m$ be a minimal prime ideal containing x. Use without proof that $\mathrm{ht}(P_x) = 1$ if $x \in m \setminus \{0\}$. Show that:

c) For any $x \in m$, such a P_x always exists.

d) $\dim(R) = \dim(R/P_x) + 1$

e) $\dim(R) \leq \mu(m)$

Exercise 2 (5 points)
Let R be a noetherian, integral domain which is not a field. Show that the following statements are equivalent:

(i) R is a discrete valuation ring.

(ii) For all $x \in \mathrm{Quot}(R)$ we have $x \in R$ or $x^{-1} \in R$.

(iii) The set of all principal ideals of R is totally ordered.

(iv) The set of all ideals of R is totally ordered.

(v) R is a local ring and a principal ideal domain.

Exercise 3 (2 points)
Show that a sequence $(a_n)_{n\in\mathbb{N}} \subseteq (\mathbb{Q}, |\cdot|_p)$ is Cauchy if and only if $|a_{n+1} - a_n|_p \stackrel{n\to\infty}{\longrightarrow} 0$.

Exercise 4 (4 points)
Let R be a discrete valuation ring with maximal ideal m. Let

$$S := \{(x, y) \in R \times R \mid x - y \in m\}.$$

Show that:

a) S is a subring of $R \times R$.

b) S is local.

c) There are exactly three prime ideals in S, i.e. two prime ideals which are not maximal.

Solutions to be handed in on Tuesday, 8. 7. 2008, at the beginning of the problem session in S12.