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Algebra II – Problem Sheet 13

Exercise 1 (6 points)
Let p be a prime number and $m \in \mathbb{N}_{\geq 1}$ be not divisible by p. Show that:

a) If $\zeta = 1 + \sum_{i=1}^{\infty} a_i p^i \in \mathbb{Z}_p$ with $\zeta^m = 1$ then $\zeta = 1$.

b) There is a primitive m-th root of unity in $\mathbb{Q}_p$ if and only if m is a divisor of $p - 1$.

Exercise 2 (4 points)
Given rings R_i ($i \in \mathbb{N}$) and homomorphisms $\varphi_i : R_{i+1} \to R_i$ ($i \in \mathbb{N}$) we define

$$\hat{R} := \{(x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} R_i \mid \forall\, i \in \mathbb{N} : \ \varphi_i(x_{i+1}) = x_i\}.$$

A tuple $(S, (g_i)_{i\in\mathbb{N}})$ consisting of a ring S and homomorphisms $g_i : S \to R_i$ ($i \in \mathbb{N}$) is called a *pferft* if $\varphi_i \circ g_{i+1} = g_i$ for all $i \in \mathbb{N}$.

a) Show that there exist homomorphisms $f_i : \hat{R} \to R_i$ ($i \in \mathbb{N}$) such that $(\hat{R}, (f_i)_{i\in\mathbb{N}})$ is a pferft with the following universal property:

   For any pferft $(S, (g_i)_{i\in\mathbb{N}})$ there is a unique homomorphism
   $\Phi : S \to \hat{R}$ such that $f_i \circ \Phi = g_i$ for all $i \in \mathbb{N}$.

b) Find the connection to $\mathbb{Z}_p$.

Exercise 3 (8 points)
A map $\|\cdot\| : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ is called a *valuation* on $\mathbb{Q}$ if the following properties hold:

$$\begin{array}{ll} \forall\, x \in \mathbb{Q}: & \|x\| = 0 \Leftrightarrow x = 0 \\ \forall\, x, y \in \mathbb{Q}: & \|xy\| = \|x\| \cdot \|y\| \\ \forall\, x, y \in \mathbb{Q}: & \|x + y\| \leq \|x\| + \|y\| \end{array}$$

A valuation $\|\cdot\|$ is called *non-archimedean* if $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathbb{Q}$. Every valuation $\|\cdot\|$ defines a metric on $\mathbb{Q}$ via $d(x, y) := \|x - y\|$ and hence a topology on $\mathbb{Q}$. Two valuations on $\mathbb{Q}$ are called *equivalent* if they define the same topology on $\mathbb{Q}$. In this exercise we want to prove the Theorem of Ostrowski which says that every valuation on $\mathbb{Q}$ is equivalent to one of the following valuations:

(i) the absolute value $|\cdot|$,
(ii) the p-adic valuation $|\cdot|_p$ for some prime number p,
(iii) the trivial valuation $|x|_{\text{triv}} := 1$ for $x \in \mathbb{Q} \setminus \{0\}$ and $|0|_{\text{triv}} := 0$.

a) Show that two valuations $\|\cdot\|_1, \|\cdot\|_2$ on $\mathbb{Q}$ are equivalent if there exists some $\alpha \in \mathbb{R}_{>0}$ such that for all $x \in \mathbb{Q}$ we have $\|x\|_1 = \|x\|_2^\alpha$.

b) Let $\|\cdot\|$ be a non-archimedean, nontrivial valuation on $\mathbb{Q}$. Show that:
   - There is a minimal prime number p with $\|p\| < 1$.
   - For every $n \in \mathbb{Z}$ not divisible by p we have $\|n\| = 1$.
   - $\|\cdot\|$ is equivalent to the p-adic valuation $|\cdot|_p$.

c) Show that any valuation $\|\cdot\|$ on $\mathbb{Q}$ with $\|n\| \leq 1$ for all $n \in \mathbb{N}$ is non-archimedean.
   Hint: Show that for all $x \in \mathbb{Q}$, $m \in \mathbb{N}_{\geq 1}$ we have $\|x+1\|^m \leq (m+1) \cdot \max\{\|x\|^m, 1\}$.

d) Let $\|\cdot\|$ be a valuation on $\mathbb{Q}$ which is not non-archimedean. Show that:
   - There is a minimal $n_0 \in \mathbb{N}$ with $\|n_0\| > 1$ and an $\alpha \in \mathbb{R}_{>0}$ with $\|n_0\| = n_0^\alpha$.
   - There is a $c \in \mathbb{R}_{>0}$ such that $\|n\| \leq cn^\alpha$ for all $n \in \mathbb{N}$.
   - We have $\|n\| \leq n^\alpha$ for all $n \in \mathbb{N}$.
   - There is a $c' \in \mathbb{R}_{>0}$ such that $\|n\| \geq c'n^\alpha$ for all $n \in \mathbb{N}$.
   - We have $\|n\| \geq n^\alpha$ for all $n \in \mathbb{N}$.
   - $\|\cdot\|$ is equivalent to the absolute value $|\cdot|$.

   Hint: Any $n \in \mathbb{N}$ can be written in the form $n = a_0 + a_1 n_0 + \ldots + a_r n_0^r$ for some $a_0, \ldots, a_r \in \{0, \ldots, n_0 - 1\}$.

Solutions to be handed in on Tuesday, 15. 7. 2008, at the beginning of the problem session in S12.