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Algebra – problem sheet 1

Exercise 1 (4 points)

In this exercise we will show that $\mathrm{PSL}_2(\mathbb{C}) := \mathrm{SL}_2(\mathbb{C})/\{\pm I_2\}$ is a simple group. First we show that any normal subgroup N of $\mathrm{SL}_2(\mathbb{C})$ with $\{\pm I_2\} \subsetneq N$ is the whole group $\mathrm{SL}_2(\mathbb{C})$.

a) Show that $\mathrm{SL}_2(\mathbb{C})$ is generated by all matrices $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ with $* \in \mathbb{C}$.

b) Let $A \in \mathrm{SL}_2(\mathbb{C})$. Show that there is an $S \in \mathrm{SL}_2(\mathbb{C})$ with $S^{-1}AS$ being in Jordan canonical form.

c) Now let $A \in N, A \neq \pm I_2$. Consider two cases for the Jordan canonical form. Show that in both cases all the matrices given in a) are contained in N. Conclude that $N = \mathrm{SL}_2(\mathbb{C})$.

d) Show that $\mathrm{PSL}_2(\mathbb{C})$ is a simple group.

Exercise 2 (6 points)

Let $p \in \mathbb{P}$ be any fixed prime number, $n \in \mathbb{N}$. You may use the following statements from EAZ:

- We have $\mathbb{F}_{p^d} \subseteq \mathbb{F}_{p^n} \Leftrightarrow d|n$.
- For $\alpha \in \mathbb{F}_{p^n}$ there is a (unique) monic[1], irreducible polynomial $m_\alpha \in \mathbb{F}_p[X]$ (called the *minimal polynomial*) with $m(\alpha) = 0$.
- We find $\deg(m_\alpha)|n$. It holds $\deg(m_\alpha) = d \Rightarrow \alpha \in \mathbb{F}_{p^d} \Leftrightarrow \alpha^{(p^d)} = \alpha$.
- Vice versa any monic, irreducible polynomial in $\mathbb{F}_p[X]$ of degree d is the minimal polynomial of an element $\alpha \in \mathbb{F}_{p^d}$.

Now let $a_n := \#\{f \in \mathbb{F}_p[X] : f \text{ is monic, irreducible and of degree } n\}$. In this exercise we want to calculate a_n.

a) Show that $x \mapsto x^p$ is a ring automorphism of $\mathbb{F}_{p^n}$. (This automorphism is called the *Frobenius automorphism.*)

b) Let $f \in \mathbb{F}_p[X]$. Let $\alpha \in \mathbb{F}_{p^n}$ be a zero of f. Show that $\alpha^{(p^k)}$ is a zero of f for all $k \in \mathbb{N}_0$.

c) Let f from b) be irreducible, monic and of degree d. Show that $f = \prod_{i=0}^{d-1} (X - \alpha^{(p^i)})$.

d) Show that the maps $n \mapsto p^n$ and $\eta * (d \mapsto d \cdot a_d)$ (both going from $\mathbb{N} \to \mathbb{N}$) are the same arithmetic functions.

e) Conclude that $n \cdot a_n = \sum_{d|n} \mu(\frac{n}{d}) \cdot p^d$ for all $n \in \mathbb{N}$.

f) Calculate a_n for $n = 2, 3, 4, 6$.

In this exercise η is given by $\eta(n) = 1$ for all $n \in \mathbb{N}$ and μ means its inverse, the Möbius function.

[1] This means that the leading coefficient is 1.

There are no more exercises on this side. Your teaching assistant uses the free space to apologize in advance for his bad English.

Return your solutions until wednesday, October 26th 2011, 7:45 in the yellow box labeled 'Algebra', Allianzgebäude, 1C or bring them directly to the problem class, 8:00.