Karlsruher Institut für Technologie
Institut für Algebra und Geometrie
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Algebra – Problem Sheet 11

Let $f \in \mathbb{C}[X]$ be a monic polynomial of degree n wih zeroes $z_1, ..., z_n \in \mathbb{C}$. The *discriminant* $D(f)$ *of* f is defined as $D(f) = \prod_{i>j} (z_i - z_j)^2$. It is independent of the order of the zeroes.

Exercise 1 (4 points)
Let $a \in \mathbb{C}$ be any zero of $f = X^3 - 2 \in \mathbb{Q}[X]$. Concretely calculate the following terms:
The discriminant of the order $\mathbb{Z}[a]$ of $\mathbb{Q}(a)$ and the discriminant $D(f)$ (as defined above).

Exercise 2 (4 points)
Now we want to prove the fact from exercise 1 (which we hopefully saw there) in general.
Let $\mathbb{Q} \subseteq K$ be a finite field-extension and $\alpha \in \mathcal{O}_K$ an integral element which generates an order in K. (In exercise 1 we have seen a concrete example for this condition.)

Let $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ be the different roots of the minimal polynomial f of α.
Use the isomorphism $\mathbb{Q}[X]/(f) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}[X]/(f)$ and the Chinese remainder theorem to find an isomorphism between $K \otimes_{\mathbb{Q}} \mathbb{C}$ and $\mathbb{C}^n$.
Calculate the traces on the $\mathbb{C}^n$ side and find the discriminant of $\mathbb{Z}[\alpha]$ via basis transformation. Knowing the determinant of the Vandermonde-matrix may help!

Exercise 3 (4 points)
Let $K \subset L$ be a finite separable extension of fields. Show that

a) The trace map extends to an L-linear map from $L \otimes_K L$ to L.

b) $L \otimes_K L$ is isomorphic to $L \times A$ for some finite-dimensional L-algebra A.

c) Use a) and b) to show that the trace map is non-zero[1].

Hand in your solutions until wednesday, January 18th 2012, 7:45 in the yellow box labeled „Algebra", Allianzgebäude, 1C or bring them directly to the problem class, 8:00.

[1]Note that the direct proof $\mathrm{tr}(1) = [L : K]$ only works in case of characteristic $\neq 0$ as $[L : K]$ may be 0 in K.