Karlsruher Institut für Technologie
Institut für Algebra und Geometrie
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Algebra – Problem Sheet 12

On this sheet every ring is commutative.

Exercise 1 (3 points)
Let S be a ring and $G \leq \mathrm{Aut}(S)$ such that all orbits $Gs, s \in S$ are finite.
Show that the fixed ring S^G is a subring of S and that the ring extension is integral.

Exercise 2 (5 points)
Let $R \subseteq S$ be an integral extension.

a) If S is a field, then R also is a field.

b) For every maximal ideal $M \subset S$, $M \cap R$ is a maximal ideal in R.

c) Let $S = \mathbb{C}[X,Y]/(Y^2 - X^3 + 1)$. Give a bijection between the set of all maximal ideals in S and the set
$$\{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3 - 1\}.$$
Hint: Use suitable rings R for which you know the maximal ideals!

Exercise 3 (5 points)
In this exercise we are going to (partly) proof a fact about chains of prime ideals in rings R, S where $R \subseteq S$ is an integral ring extension. Let I be an ideal in S. Prove the following facts:

a) $R \cap I$ is an ideal in R.

b) If I is a prime ideal in S the ideal $R \cap I$ is prime in R.

c) If $I \neq 0$ then $R \cap I \neq 0$.

d) For any (finite or infinite) chain of different prime ideals in R we get a prime ideal chain in S of the same length.

Addendum: Without proof you may note and use the fact that the other direction works, too. If you have a (prime) ideal $J \in R$ you are able to lift it to a (prime) ideal of S:
$S \cdot J := \{ \sum_i^{<\infty} s_i j_i : s_i \in S, j_i \in J\}$.
$S \cdot J$ satisfies $(S \cdot J) \cap R = J$. Any chain of prime ideals in R gives rise to a chain of prime ideals of the same length.

Exercise 4 (3 points)
For any ring R define its *Krull dimension* $\dim(R) \in \mathbb{N} \cup \{\infty\}$ to be the length of the longest chain $\mathfrak{p}_1 \subsetneqq \mathfrak{p}_2 \subsetneqq \ldots$ of prime ideals. The Krull dimension may be infinite.

a) Give an example of a ring R that has infinite Krull dimension.

b) Prove that R, S have the same Krull dimension if $R \subseteq S$ is an integral ring extension.

c) Find rings $R \subseteq S$ such that $\dim(R) < \dim(S)$.

d) Find rings $R \subseteq S$ such that $\dim(S) < \dim(R)$.

Hand in your solutions until wednesday, January 25th 2012, 7:45 in the yellow box labeled „Algebra“, Allianzgebäude, 1C or bring them directly to the problem class, 8:00.