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Algebra – Problem Sheet 13

Exercise 1 (3 points)
Let R be a Dedekind domain and $I, J \subseteq R$ two non-zero ideals. Let furthermore I and J be coprime, i.e. $I + J = R$.
Show that $I \cap J = IJ$.
Give an example that you need coprime for this statement.

Exercise 2 (5 points)
Let D be a Dedekind domain with quotient field K.

a) Let P be a prime ideal of D. Show that $P^2 \subsetneq P$.

b) Let $P_1, ..., P_r$ be pairwise different prime ideals, $e_1, ..., e_r \in \mathbb{Z}$. Show that there is an $r \in K$ such that $v_{P_i}(x) = e_i$ for all $i = 1, ..., r$.
(*Hint:* If $x = \frac{a}{b}$ nominator and denominator may be responsible for different types of e_i. Use the Chinese Remainder theorem to construct adequate a, b. Use a)...)

c) Now let D have only finitely many prime ideals. Show that D is a prinicipal ideal domain.

Exercise 3[1] (8 points)
Let $R = \mathbb{Z}[\sqrt{-5}]$ be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{-5})$. By "ideal" we will always mean non-zero ideal in R. The class of an ideal is its class in the class group. Show the following statements:

a) There exists exactly one ideal T of norm 2. This is not generated by one element, but T^2 is a principal ideal.

b) All ideals of index less than 6 are either principal or in the class of T.

c) For every prime number p there exists an ideal of norm p if and only if there exists an integer $b \in \mathbb{Z}$ such that p divides $b^2 + 5$.

d) For each of the prime numbers in c) there exists an element $r \in R$ such that $N(r) \in \{p, 2p, 3p, 4p, 5p\}$.
Hint: Let b be as in c). There are more than p pairs of integers $0 \leq z \leq \sqrt{p}$. Conclude that in Dirichlet's pigeon hole[2] there are two pairs (x, y) and (x', y') of such integers such that $x - by \equiv x' - by' \pmod{p}$. Estimate the norm of $(x - x') + (y - y')\sqrt{-5}$!

e) For every prime ideal $P \subset R$ either P or TP is a principal ideal.

f) The class-group of R consists of two elements.

g) The prime numbers satisfying the criterion from c) are exactly 2, 5, and the prime numbers congruent to $1, 3, 9$ or 7 modulo 20.
For each of these prime numbers p, either p or $2p$ is a norm in R.

Hand in your solutions until wednesday, February 1st 2012, 7:45 in the yellow box labeled „Algebra", Allianzgebäude, 1C or bring them directly to the problem class, 8:00.

[1]Looking at example 4.2.9 may be helpful.
[2]„Schubfachprinzip"