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Institut für Algebra und Geometrie
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Algebra – Problem Sheet 14 - the last sheet[1]

Exercise 1 (4 points)
Let K be a field with non-archimedean valuation $|\cdot|$.

a) $K_r(x) = \{y \in K : |y - x| \leq r\}$ is a circle. Then for any $y \in K_r(x)$ we have $K_r(x) = K_r(y)$, meaning any point of the circle is middle point.

b) Prove that any triangle in K is an isosceles triangle, where the third side is at most as long as the two long sides of equal length. (Any triangle is given by three (not necessarily different) points. Consider the distances between three arbitrary points.)

Aufgabe 2 (6 points)
Let K be the quotient field of $R := \mathbb{C}[X, Y]/(Y^2 - X^3 + 1)$ (compare exercise 2 on problem sheet 12) and let

$$E := \{(x, y) \in \mathbb{C}^2 \mid y^2 = x^3 - 1\} \cup \{\infty\},$$

where $\infty \notin \mathbb{C}^2$.
A discrete valuation on K is called *normalized*, if its value group is $\mathbb{Z}$. Every non-trivial discrete valuation is equivalent to a normalized one.

a) For which normalized valuation on K does (the class of) X not belong to the valuation ring? What is the valuation ring in this case? Find a generator of the valuation ideal.

b) Give a bijection between the set of normalized valuations on K and E.

c) Let $a \in \mathbb{C}$ be arbitrary. Write down all the valuations of (the class of) $X - a$ for your valuations from b). What is the sum of these values?

Exercise 3 (4 points)
Now let $p \geq 3$ be an odd prime number.
Prove that $\mathbb{Q}_p$ contains a primitive $(p-1)$-th root of unity but 1 is the only p-th root of unity.
(*Hint:* Note that any such root is an element of $\mathbb{Z}_p$ and remember Hensel's lemma.)

Hand in your solutions until wednesday, February 8th 2012, 7:45 in the yellow box labeled „Algebra", Allianzgebäude, 1C or bring them directly to the problem class, 8:00.

[1]Yes, it is true. Thanks for many wednesday mornings and good luck in the examinations.