Karlsruher Institut für Technologie
Institut für Algebra und Geometrie
PD Dr. Stefan Kühnlein Dipl.-Math. Jochen Schröder

Algebra – Problem Sheet 3

Exercise 1 (2 points)
Let R be a commutative ring. R is called *a local ring* if it contains only one maximal ideal.
Show that R is local if and only if $R \setminus R^\times$ is an ideal.

Exercise 2 (4 points)
Let $R = C[0,1]$ be the ring of all continuous functions from $I = [0,1]$ to $\mathbb{R}$.

a) Let $i \in I$. Show that $m_i := \{f \in R : f(i) = 0\}$ is a maximal ideal in R.

b) Show that any proper ideal in R is contained in an m_i.

c) Conclude that $\{m_i : i \in I\} = \{m : m \text{ maximal ideal in } R\}$.

Exercise 3 (2 points)
Let $K \subseteq L$ be a field extension and $\alpha \in L$. $K[\alpha]$ is defined as the smallest subring of L containing K and α, while $K(\alpha)$ is the smallest subfield of L containing K and α.
Show that $K[\alpha] = K(\alpha)$ if and only if α is algebraic over K.

Exercise 4 (4 points)
Let K be a field and $f, g \in K[X]$ two irreducible polynomials with coprime degrees. Now let α be a zero of f in a field extension L of K. (W.l.o.g. you can assume that L also contains a zero β of g (why?)).
Show that g is irreducible over $K(\alpha)$.

Exercise 5 (4 points)

a) Let $n \in \mathbb{N}$. Show that $\sqrt{n} \in \mathbb{Q} \Leftrightarrow \sqrt{n} \in \mathbb{N}$.

b) Calculate $[\mathbb{Q}(\sqrt{n}, \sqrt{n+1}) : \mathbb{Q}]$ in dependence on n.

c) Calculate $[\mathbb{Q}(\sqrt{n} + \sqrt{n+1}) : \mathbb{Q}]$ by finding the minimal polynomial of $\sqrt{n} + \sqrt{n+1}$.

d) Conclude that $\sqrt{n}$ is contained in $\mathbb{Q}(\sqrt{n} + \sqrt{n+1})$.

The teaching assistant has been convinced that we can save paper by putting the hand-in-line on the front side. So this time there is absolutely nothing on the back side. Please don't print it! Thank you.

Hand in your solutions until wednesday, November 9th 2011, 7:45 in the yellow box labeled „Algebra", Allianzgebäude, 1C or bring them directly to the problem class, 8:00.