Karlsruher Institut für Technologie
Institut für Algebra und Geometrie
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Algebra – Problem Sheet 4

Exercise 1 (4 points)
Let K be a field. Show one of the following statements and use it to show the other one:

a1) There are infinitely many monic, irreducible polynomials in $K[X]$.

1a) If K is finite K is not algebraically closed.

Exercise 2 (6 points)
Show that the following polynomials are irreducible.

a) $X^{p-1} + ... + X + 1 \in \mathbb{Q}[X]$ for $p \in \mathbb{P}$ (*hint:* The coordinate change $X \mapsto X + 1$ may help you.)

b) $\Phi_{p^d}(X) \in \mathbb{Q}[X]$ for $p \in \mathbb{P}, d \in \mathbb{N}$ (where Φ_n is the n-th cyclotomic polynomial)

c) $X^3 - Y^2X^2 + YX + 4Y \in \mathbb{C}[X, Y]$

d) $Y^2 - X^3 \in \mathbb{C}[X, Y]$

Exercise 3 (6 points)
Consider $f = Y^2 - X^3 + 1 \in \mathbb{C}[X, Y]$. Recall its irreducibility from the lecture.
Sketch the zero set $\{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$.

Let $E := \{(x, y) \in \mathbb{C}^2 : y^2 = x^3 - 1\}$ be the zero set of f in $\mathbb{C}^2$ and let $I = \{g \in \mathbb{C}[X, Y] : g|_E = 0\}$ be the ideal of all polynomials that vanish on E.
Show that f divides $g \in \mathbb{C}[X, Y]$ if and only if g has infinitely many zeroes in E.
(*Hint for „⇐“:* For any g with this property begin considering g modulo f. Try to reduce the degree of y. Now try to eliminate all y and prove the claim by using that g has infinitely many zeroes in E. What does this mean considering only the first component?)

Conclude that $I = (f)$. Furthermore conclude that this ideal is a prime ideal.
Finally calculate the transcendence degree for the field extension $\mathbb{C} \subseteq \mathrm{Quot}(\mathbb{C}[X, Y]/I)$.

Hand in your solutions until wednesday, November 16th 2011, 7:45 in the yellow box labeled „Algebra“, Allianzgebäude, 1C or bring them directly to the problem class, 8:00.