Karlsruher Institut für Technologie
Institut für Algebra und Geometrie
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Algebra – Problem Sheet 5

Exercise 1 (4 points)
Let $\zeta \in \mathbb{C}$ be any primitive seventh root of unity, that is $\zeta^7 = 1$ but $\zeta^i \neq 1$ for $1 \leq i \leq 6$. Show the following statements:

a) $\sum_{i=0}^{6} \zeta^i = 0$.

b) $\zeta \cdot \overline{\zeta} = 1$. Conclude that $\mathbb{Q} \subseteq \mathbb{Q}(\zeta + \overline{\zeta}) \subseteq \mathbb{Q}(\zeta)$.

c) What is $[\mathbb{Q}(\zeta) : \mathbb{Q}]$?

d) Find the minimal polynomial of $\zeta + \overline{\zeta}$ over $\mathbb{Q}$ and the minimal polynomial of ζ over $\mathbb{Q}(\zeta + \overline{\zeta})$.

e) Prove that $\mathbb{Q}(\zeta + \overline{\zeta})$ is a galois field extension over $\mathbb{Q}$.

Exercise 2 (4 points)
Let $p \in \mathbb{P}$ be any prime number.

a) Let K be a field of characteristic p. Let $\sigma : K \to K, x \mapsto x^p$ be the Frobenius homomorphism. Show that K is perfect if and only if σ is surjective.

b) Find a field K with characteristic p that is not perfect. Furthermore find an infinite perfect field K with characteristic p.

Exercise 3 (4 points)
For $n \in \mathbb{N}$ let $f = X^n - 2 \in \mathbb{Q}[X]$ and α be any zero of f.
Show that $\mathbb{Q}(\alpha)$ is a normal field extension of $\mathbb{Q}$ if and only if $n = 1$ or $n = 2$.

Exercise 4 (4 points)
Let $\alpha = i + \sqrt{\frac{1}{2}} \in \mathbb{C}$. Show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ and that the field extension $\mathbb{Q} \subseteq \mathbb{Q}(\alpha)$ is a galois field extension.
Find a basis B of $\mathbb{Q}(\alpha)$ as a $\mathbb{Q}$ vector space. Give the transformation matrix D_{BB} for any automorphism in the galois group $\mathrm{Gal}(\mathbb{Q}(\alpha)|\mathbb{Q})$.
Finally find a subgroup of S_4 that is isomorphic to $\mathrm{Gal}(\mathbb{Q}(\alpha)|\mathbb{Q})$.

Hand in your solutions until wednesday, November 23rd 2011, 7:45 in the yellow box labeled „Algebra“, Allianzgebäude, 1C or bring them directly to the problem class, 8:00.