Karlsruher Institut für Technologie
Institut für Algebra und Geometrie
PD Dr. Stefan Kühnlein   Dipl.-Math. Jochen Schröder

Algebra – Problem Sheet 6

exercise 1 (6 points)

a) Consider the following polynomials in $\mathbb{Q}[X]$. Find their splitting field L and the Galois group $\mathrm{Gal}(L|\mathbb{Q})$.

1. $X^4 + X^2 + 1$
2. $(X^3 - 2)(X^2 - 3)$
3. $X^4 - 2$

b) Now find $n \geq 4$ and a polynomial $f \in \mathbb{Q}[X]$ of degree n, such that the Galois group of its splitting field over $\mathbb{Q}$ is isomorphic to S_n. (Justify your answer.)

exercise 2 (5 points)
Let k be a field, $R = k[X]$ and $K = \mathrm{Quot}(R)$. The map $f(X) \mapsto f(X+1)$ is an automorphism of R and induces an automorphism σ of K:

$$\forall f(X), g(X) \in R,\ g \neq 0 : \sigma(\frac{f(X)}{g(X)}) = \frac{f(X+1)}{g(X+1)}.$$

a) σ has finite order if and only if $\mathrm{char}(k) \neq 0$.

b) For $\mathrm{char}(k) = 0$ the fixed field K^σ is k.

c) Calculate K^σ in the case $\mathrm{char}(k) = p \neq 0$. Is $K^\sigma \subseteq K$ a Galois extension in this case? Are K and K^σ isomorphic?

exercise 3 (4 points)

a) Calculate $\mathrm{Aut}(\mathbb{R}|\mathbb{Q})$. (*Hint:* Use that the square of any real number is non-negative to show that any automorphism of $\mathbb{R}$ is monotone).

b) Now show that there is no proper, finite, normal field extension $K \subsetneq \mathbb{R}$.

c) Conclude that there is no subfield K of $\mathbb{R}$ with $[\mathbb{R} : K] = 2$.

Hand in your solutions until wednesday, November 30th 2011, 7:45 in the yellow box labeled „Algebra“, Allianzgebäude, 1C or bring them directly to the problem class, 8:00.