Karlsruher Institut für Technologie
Institut für Algebra und Geometrie
PD Dr. Stefan Kühnlein   Dipl.-Math. Jochen Schröder

Algebra – Problem Sheet 7

Exercise 1 (3 points)
Many years ago a cruel pest epidemic came to the island Delos. The Delians went to the oracle of Delphi and asked for advice. "You need to construct a stone cube double the volume of the big stone cube in the temple of Apoll" was the answer, "then you will be disburdened from the pest."
The Delians were unable to do anything but compass and straight-edge constructions. Were they able to double the size as ordered? (Don't be too sad, the legend tells us that in the end Apoll was satisfied with an approximate solution.) Would the Delians have been able to solve the problem if instead of volume they should have doubled the surface area?

Exercise 2 (5 points)
Let $\zeta = \exp(\frac{2\pi i}{15})$. Determine all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$ and give a construction of the regular 15-gon with compass and straight-edge construction.

Exercise 3 (8 points)
Let K be a field. In the lecture we have seen that any finite separable field extension $K \subseteq L$ is a simple one, meaning there exists a primitive element $\alpha \in L$, such that $L = K(\alpha)$. We want to see another more constructive proof here.

a) First answer the following questions: Why can we assume w.l.o.g. that K is infinite? Why can we furthermore assume that $L = K(\alpha, \beta)$ is generated by two elements?

Now let $L = K(\alpha, \beta)$ be a seperable finite extension of the infinite field K of degree $[L : K] = n$.

b) Show that there are n K-homomorphisms $\mathrm{Hom}_K(L, \overline{K}) = \{\sigma_1, ..., \sigma_n\}$ from L to the algebraic closure $\overline{K}$.

Define $g(X) = \prod_{1 \le i < j \le n} \Big(\big(\sigma_i(\alpha) - \sigma_j(\alpha)\big) + \big(\sigma_i(\beta) - \sigma_j(\beta)\big) \cdot X \Big)$.

c) Show that $g \not\equiv 0$.

Choose any $\lambda \in K$ such that $g(\lambda) \neq 0$. Let $f \in K[X]$ be the minimal polynomial of $\gamma := \alpha + \lambda\beta$. Show the following statements for γ.

d) $f(\sigma(\gamma)) = 0$ for any $\sigma \in \mathrm{Hom}_K(L, \overline{K})$.

e) $\sigma_i(\gamma) \neq \sigma_j(\gamma)$ for $i \neq j$.

f) γ is a primitive element as demanded.

Let now L be the splitting field of $X^3 - 2$ over $\mathbb{Q}$. Remember that it is a field extension of degree 6 which we have discussed in several cases over the last weeks.

g) Use the construction to find a primitive element of L over $\mathbb{Q}$. For $\alpha = \sqrt[3]{2}, \beta = \zeta := \exp(\frac{2\pi i}{3})$ find the polynomial g, an adequate λ and a primitive γ.

Hand in your solutions until wednesday, December 7th 2011, 7:45 in the yellow box labeled „Algebra", Allianzgebäude, 1C or bring them directly to the problem class, 8:00.