Karlsruher Institut für Technologie
Institut für Algebra und Geometrie
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Algebra – Problem Sheet 9

On this sheet any ring is commutative and has a 1.

Exercise 1 (4 points)
In the lecture we have learned that a ring R is noetherian if and only if any ascending chain of ideals becomes stationary. Analogously we call R *artinean* if this holds for any descending chain: For any chain $(I_n)_{n\in\mathbb{N}}$ with ideals I_n such that $I_{n+1} \subseteq I_n$ for any $n \in \mathbb{N}$ we'll find an index n_0 such that $I_m = I_{n_0}$ for all $m \geq n_0$.

a) Show that an artinean ring is noetherian.

b) Find a counterexample to the opposite direction.

Exercise 2 (4 points)
Remember the proof that for any noetherian ring R the polynomial ring $R[X]$ is noetherian as well. Try to vary the proof to show that the statement also holds for the ring $R[[X]]$ of formal power series instead of the polynomial ring. (*Hint:* Find out why you can't directly take the proof from the lecture. Considering the lowest power of X that has not 0 as coefficient instead of the degree may be helpful.)

Exercise 3 (4 points)
Show that the ring $\{f \in \mathbb{Q}[X] : f(0) \in \mathbb{Z}\}$ of rational polynomials with integer constant term is not a noetherian ring.

Exercise 4 (4 points)
Let R, S be two rings.

a) Show that the product $R \times S$ is noetherian if and only if R and S are noetherian.

b) Now let R, S be noetherian and T be a subring of $R \times S$. Show that T is noetherian[1] if both projections $T \to R$, $T \to S$ are surjective.
Find an example which shows that this is not an equivalence.

Hand in your solutions until wednesday, December 21st 2011, 7:45 in the yellow box labeled „Algebra“, Allianzgebäude, 1C or bring them directly to the problem class, 8:00.

[1] Note that this is not trivial. In exercise 3 we see a non-noetherian subring of a noetherian ring.