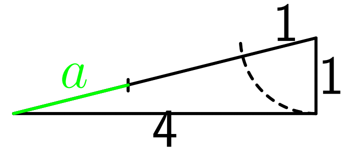
A construction of the regular 17-gon

Let $z = \exp(2\pi i/17)$.
Vertices of the regular 17-gon: $1, z, z^2, \ldots, z^{16}$.
We have to construct z. With $K := \mathbb{Q}(z)$ we have $[K : \mathbb{Q}] = 16$.

$$0 = \frac{z^{17}-1}{z-1} = 1 + z + z^2 + \cdots + z^{16}.$$

$G = \mathrm{Gal}(K|\mathbb{Q})$ is cyclic of order 16. One possible generator is $\sigma : z^m \mapsto z^{6m}$ $(0 \leq m \leq 16)$.
Construction of intermediate extensions:
There is one subfield of K of degree 2,4, und 8 each over $\mathbb{Q}$. They are generated by
$a := z + z^2 + z^4 + z^8 + z^9 + z^{13} + z^{15} + z^{16}$ (invariant under σ^2),
$b := z + z^4 + z^{13} + z^{16}$ (invariant under σ^4), and
$c := z + z^{16}$. We have $c = 2\cos(2\pi/17)$.
Constructions for a, b, and c:

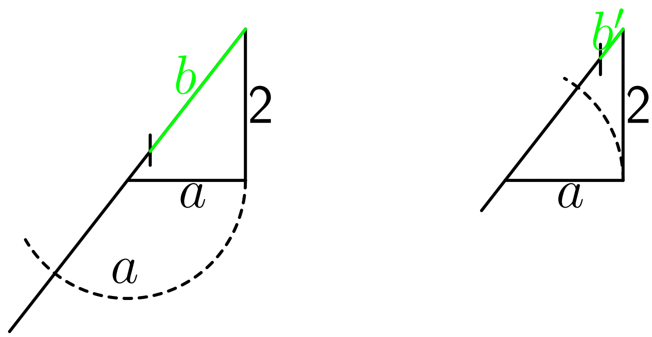
First Step: Construction of a : $a = \frac{\sqrt{17}-1}{2}$.
As $17 = 4^2 + 1^2$ we may invoke Pythagoras:



Second Step: Construct b.
The other zero of the minimal polynomial of b over $\mathbb{Q}(a)$ is $\sigma^2(b) = z^2 + z^8 + z^9 + z^{15}$. We have

$$(X - b)(X - \sigma^2(b)) = X^2 - aX - 1,$$

hence $b = \frac{a+\sqrt{a^2+4}}{2}$ (plus, as b is positive). We also need $b' := -\sigma^2(b) = \frac{\sqrt{a^2+4}-a}{2}$. The numbers b and b' may be constructed that way:



Third step: Construction of $c = z + z^{-1}$.

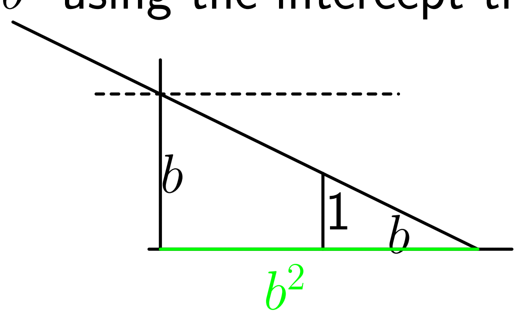
The second zero of the minimal polynomial of c over $\mathbb{Q}(b)$ is $\sigma^4(c) = z^4 + z^{-4}$. We find

$$\begin{aligned}(X-c)(X-\sigma^4(c)) &= X^2 - bX + (z^3 + z^5 + z^{12} + z^{14}) \\ &= X^2 - bX + (b^2 - \sigma^2(b) - 4)/2.\end{aligned}$$

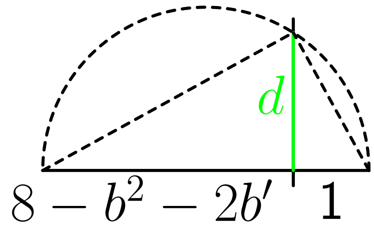
Therefore

$$c = \frac{b + \sqrt{-b^2 - 2b' + 8}}{2}.$$

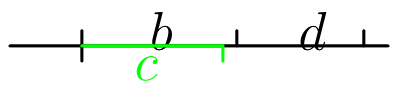
Noe we construct b^2 using the intercept theorem:



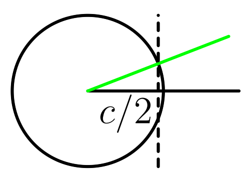
Then the Theorem on heights in rectangular triangles and Thales' Theorem help in constructing $d := \sqrt{8 - b^2 - 2b'}$:



We finally have $c = (b + d)/2$:



Because of $c/2 = \cos\frac{2\pi}{17}$ we may now cut the first piece of cake in our 17-gon as follows:



Enlarged and completed we end up with