

51	52	53	54	55	Σ

Exercise Sheet No. 11 Advanced Mathematics I

Exercise 51: Use the mean value theorem to prove the following statements:

(a) Show the inequality

$$\ln(1+x) \leq \frac{x}{\sqrt{1+x}} \quad \text{for } x > 0.$$

Hint: Consider the function $f(t) = \ln(1+t) - \frac{t}{\sqrt{1+t}}$ in the interval $[0, x]$.

(b) Show the estimates

$$1 - \frac{1}{x} < \ln x < x - 1, \quad x \in (1, \infty).$$

Can you give an upper and lower bound on the real number $a = 2 \ln 3 - 3 \ln 2$ using these inequalities?

Hint: Consider the function $f(t) = \ln t$ for $t \in (1, x)$.

(c) Show Lipschitz continuity of the function

$$f(x) = \sqrt{1+x}, \quad 0 \leq x < 3,$$

and compute a Lipschitz constant.

Exercise 52:

(a) Calculate the following limits

$$(i) \quad \lim_{x \rightarrow 0} \left[\frac{1}{e^x - 1} - \frac{1}{x} \right], \quad (ii) \quad \lim_{x \rightarrow \infty} \frac{x^2 e^x}{(e^x - 1)^2}, \quad (iii) \quad \lim_{x \rightarrow 0^+} (e^x + 3x)^{\frac{1}{x}}.$$

(b) Find a value $c \in \mathbb{R}$, such that the function $f : (0, \infty) \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{x-1} - \frac{1}{\ln(x)}, & \text{if } x \neq 1, \\ c, & \text{if } x = 1. \end{cases}$$

is continuous.

Exercise 53: Let the function $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \sin^2(x)$.

Find the Taylor polynomial of degree 5 for f at $x_0 = 0$. Use this Taylor polynomial to approximate $\sin^2(\frac{1}{10})$ and show that the error of approximation is smaller than 10^{-6} .

Exercise 54: Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \cosh \frac{x}{2}.$$

Determine all derivatives $f^{(n)}$, $n = 0, 1, 2, \dots$ of the function f and give the Taylor series for f with centre of expansion $x_0 = 0$. Where does the series converge?

Exercise 55: Let the function $f : (-1, \infty) \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{1}{x^2 + 5x + 4}$.

Find the Taylor series for f at $x_0 = 0$ and compute $f^{(5)}(0)$.

Hint: $f(x) = \frac{1/3}{x+1} - \frac{1/3}{x+4}$.

Due date: Your written solutions are due at 12:00 on Monday, **January 27, 2020**. Please submit them in the green box labelled "AM1" in the atrium of the maths building (20.30).

Website: For detailed information regarding this course visit the following web page: