Exercise Sheet No. 6
Advanced Mathematics I

Exercise 26:
Consider the polynomial \( f : \mathbb{R} \rightarrow \mathbb{R} \) given by \( f(x) := \frac{1}{8} x^3 + \frac{3}{8} x^2 - \frac{9}{8} x + \frac{5}{8} \).
(a) Expand \( f \) about the expansion points \( x_1 = 1 \) and \( x_2 = -3 \). Use this representation to discuss the behavior of \( f \) on the interval \([1, \infty)\).
(b) Use a sketch of \( f \) to find intervals on which \( f \) has an inverse function. Also sketch the inverse.

Exercise 27:
Let \( c \in \mathbb{R} \) and consider the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) given by
\[
\begin{cases}
1 - 2x - x^2, & x < 0, \\
c(x-2)^2, & x \geq 0.
\end{cases}
\]
(a) Find a value of \( c \) such that \( f \) is continuous and give a sketch of \( f(x) \) for this \( c \) on the interval \([-3, 4]\).
For the remaining exercise study \( f \) for this fixed \( c \).
(b) Find all maximal intervals \( I \) of \( \mathbb{R} \) where \( f \) is invertible.
Note: Here maximal means that there is no interval \( I' \) with \( I \subseteq I' \) and \( f \) is invertible on \( I' \).
For each of these intervals give the inverse of \( f \) and the domain of the inverse on this interval.
(c) Find a maximal domain \( D \subseteq \mathbb{R} \) such that the function \( f : D \rightarrow \mathbb{R} \) is bijective.

Exercise 28:
Show that the function \( x \mapsto \frac{1}{\sqrt{x}} \) is continuous on the domain \( D = (0, \infty) \). Sketch the graph of the function.

Exercise 29:
For each of the following functions \( f_j : \mathbb{R} \rightarrow \mathbb{R} \) find all points \( x \in \mathbb{R} \) where \( f_j \) is continuous
\[
\begin{align*}
(a) & \quad f_1(x) := \begin{cases} \\
\frac{x^3 + 4x^2 + x - 6}{x^3 - 3x + 2}, & x \in \mathbb{R} \setminus \{1, -2\}, \\
0, & x = 1, \\
-\frac{1}{3}, & x = -2,
\end{cases} \\
(b) & \quad f_2(x) := \begin{cases} x, & x \in \mathbb{Z}, \\
0, & \text{otherwise}.
\end{cases}
\end{align*}
\]

Exercise 30:
Prove that the function \( x \mapsto \sqrt[3]{x} \) is continuous on the domain \( D = [0, \infty) \).
Hint: For the case \( x \neq 0 \) use the identity \( a^3 - b^3 = (a - b)(a^2 + ab + b^2) \) for \( a, b \in \mathbb{R} \).

Due date: Your written solutions are due at 12:00 on Monday, December 2, 2019. Please submit them in the green box labelled “AM1” in the atrium of the maths building (20.30).
Problem Session: 8:00 Wednesday, November 27, 2019
Website: For detailed information regarding this course visit the following web page:
http://www.math.kit.edu/iag3/edu/am12019w/en