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Holiday sheet (without submission)
Advanced Mathematics I

Exercise 1:

(a) Determine $\lim\limits_{x \to x_0} f(x)$ for the following functions f and points x_0:

(i) $f(x) = \dfrac{x-2}{x^2-4}$ for $x > 2$, $\quad x_0 = 2$, $\qquad$ (ii) $f(x) = \dfrac{\sqrt[4]{x}-1}{\sqrt[3]{x}-1}$ for $x > 1$, $\quad x_0 = 1$.

(b) Determine all points $x \in \mathbb{R}$, at which the following function is continuous:

$$g(x) = \begin{cases} 2(x+1)^2, & x < -1, \\ -x & x \in [-1,1], \\ x^2 - 2x, & x > 1. \end{cases}$$

Exercise 2:

Consider the recursively defined sequence

$$a_1 = b\,, \qquad a_{k+1} = \frac{|a_k|}{2a_k - 1}\,, \; k \in \mathbb{N}$$

for two initial values $b = -\frac{1}{4}$, and $b = \frac{1}{4}$.

(a) Assume that for fixed b the sequence (a_k) converges. What are the candidates for the limit?

(b) Determine for which initial value $b = -\frac{1}{4}$ or $b = \frac{1}{4}$ the sequence is monotone.

(c) Determine for which initial value the sequence is bounded.

(d) Justify, for both initial values $b = -\frac{1}{4}$ or $b = \frac{1}{4}$, whether the sequence is converging or not. In case of convergence determine the limit.

Exercise 3:

Consider the polynomial p and the function f given by

$$p(x) \;=\; x^5 - 8x^2 + 4\,, \quad f(x) = |p(x)|\,, \quad x \in \mathbb{R}.$$

(a) Justify that f has a minimum point $x_- \in [-2,2]$ with minimum value $f(x_-) \leq 4$.

(b) Show that $f(x) \geq 4$ for all $|x| \geq 2$.

(c) Why does f have a minimum in $\mathbb{R}$?

Exercise 4:

Show that for any positive constants a, b, c the equation

$$\frac{(a+b)x + a - b}{x^2 - 1} + \frac{c}{x-2} = 1$$

has solutions in the intervals $[-1,1]$ and $[1,2]$, respectively.

Exercise 5:

For which $x \in \mathbb{R}$ does the following power series converge?

$$\sum_{n=1}^{\infty} (-2)^n \frac{n^2+2}{n^3+n} x^{3n}.$$

Exercise 6:

Find all solutions $z \in \mathbb{C}$ of the following equation

$$3^{-2z} + 1 = 2 \cosh(z \ln 3).$$

Exercise 7:

Determine all solutions $z \in \mathbb{C}$ of the equation

$$\left(e^{\mathrm{i}z} - 3\right) \cos(z) + \frac{3}{2} + \frac{1}{2} e^{-\mathrm{i}z} = 0.$$

Exercise 8:

The function $f : \mathbb{R} \setminus \{\frac{1}{2}\} \to \mathbb{R}$, where $f(x) = \frac{1}{1-2x}$, can be expanded into a power series centered at $x_0 = 2$.

(a) Determine the coefficients of this power series.

(b) For which $x \in \mathbb{R}$ does this series converge?

Exercise 9:

Consider the recursively given sequence

$$a_{n+1} = \frac{1}{5}(a_n^2 + 4)$$

for the initial values

$$\text{a) } a_0 = 2, \qquad \text{b) } a_0 = 4.$$

Decide whether the sequences are monotone and convergent. In case of convergence, determine the limit.

Submission: This exercise sheet will not be graded. The solutions will be uploaded in January.