Inverse:
Let $f : D \to W$ be a function. A function $g : W \to D$ is called an **inverse function** of f if

$$g(f(x)) = x \quad \text{for all } x \in D \quad \text{and} \quad f(g(y)) = y \quad \text{for all } y \in W\,. \tag{5.1}$$

The following notation is sometimes used: $g = f^{-1}$. Graphically, one finds the inverse function of a real function by reflection with respect to the line $y = x$.

The notation f^{-1} should be used with caution, not to confuse with $\frac{1}{f}$ with $\frac{1}{f}(x) = \frac{1}{f(x)} = (f(x))^{-1}$, the inversion w.r.t the multiplication.

For example, the inverse function of

$$f : [0, \infty) \to [0, \infty); f(x) = x^2$$

is given by

$$g : [0, \infty) \to [0, \infty); g(x) = \sqrt{x}.$$

Note that for $D = \mathbb{R}$ no inverse function of $f(x) = x^2$ exists since for every $y \in f(D)$ there exist two pre-images $\pm\sqrt{y}$. It is not possible to find a function g such that (5.1) holds for all $x \in \mathbb{R}$ in this case.

Theorem 5.10.

- *A function $f : A \to B$ has an inverse function if and only if it is bijective.*
- *Every injective function $f : A \to B$ can be made bijective by restricting the codomain to the range $\widetilde{f} : A \to f(A)$.*
- *In particular, a strictly monotone function $f : A \subseteq \mathbb{R} \to B \subseteq \mathbb{R}$ has an inverse (after restricting the codomain to the range).*

Proof. Let $f : A \to B$ be bijective. Define $g : B \to A$ as follows: $g(b) = a$ iff $f(a) = b$. Clearly, g is well-defined, since for each $b \in B$ there is exactly one $a \in A$ such that $f(a) = b$ as f is bijective. Now, $f(g(b)) = f(a) = b$ and $g(f(a)) = g(b)) = a$. So, g is an inverse function of f.
Now assume that f has an inverse function $g : B \to A$. Then $f(g(b)) = b$ and $g(f(a)) = a$ for any $a \in A$ and $b \in B$. Assume for the sake of contradiction that f is not a bijection. Then f is either not injective or not surjective. If f is not injective, then for some distinct $a, a' \in A$, there is $b \in B$ such that $f(a) = f(a') = b$. Then $g(f(a)) = g(b) = a$ and $g(f(a')) = g(b) = a'$, so $g(b)$ has two distinct values, a contradiction. If f in not surjective, then for some $b \in B$ there is no $a \in A$ such that $f(a) = b$. Then $f(g(b)) \neq b$, a contradiction.
Every function can be made surjective by restricting the codomain. So, if $f : A \to B$ is injective, this procedure yields a bijective function $\widetilde{f} : A \to f(A)$.
Finally, assume that $f : A \subseteq \mathbb{R} \to B \subseteq \mathbb{R}$ is strictly monotone. Since for any distinct $x, y \in A$, we have $f(x) < f(y)$ or $f(x) > f(y)$, we see that $f(x) \neq f(y)$, thus f is injective. So by the previous observations, it is bijective (after restricting the codomain) and hence has an inverse function

$$g : f(A) \subseteq \mathbb{R} \to A \subseteq \mathbb{R}.$$

□